towards stronger fairness guarantees

Rohit Vaish

The Model

Set of agents

Set of indivisible items

A B C D E

Allocation

Envy-Freeness

[Gamow and Stern, 1958; Foley, 1967]

Each agent prefers its own bundle over that of any other agent.

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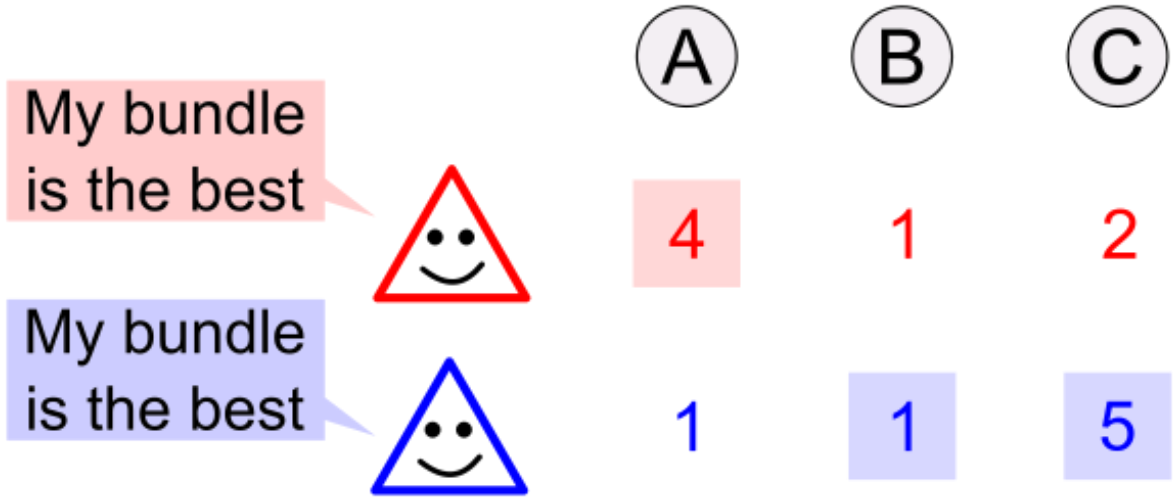

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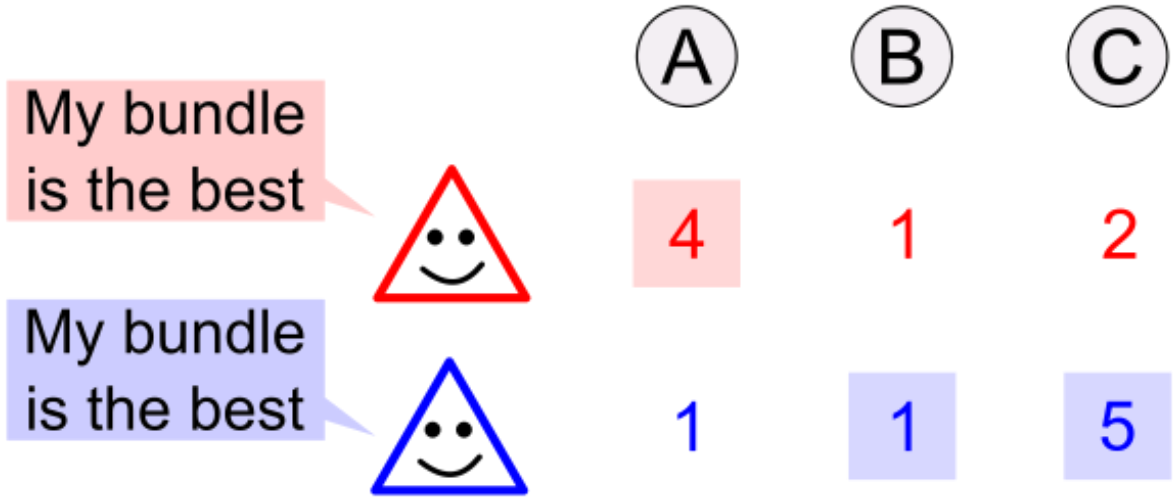

☹ Not guaranteed to exist (two agents, one good)

☹ Checking whether an EF allocation exists is NP-complete

Envy-Freeness Up To One Good

[Budish, 2011]

Envy can be eliminated by removing some good in the envied bundle.

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My bundle is better if Ⓐ is removed

My bundle is better if Ⓒ is removed

Ⓐ	Ⓑ	Ⓒ
4	1	2
1	1	5

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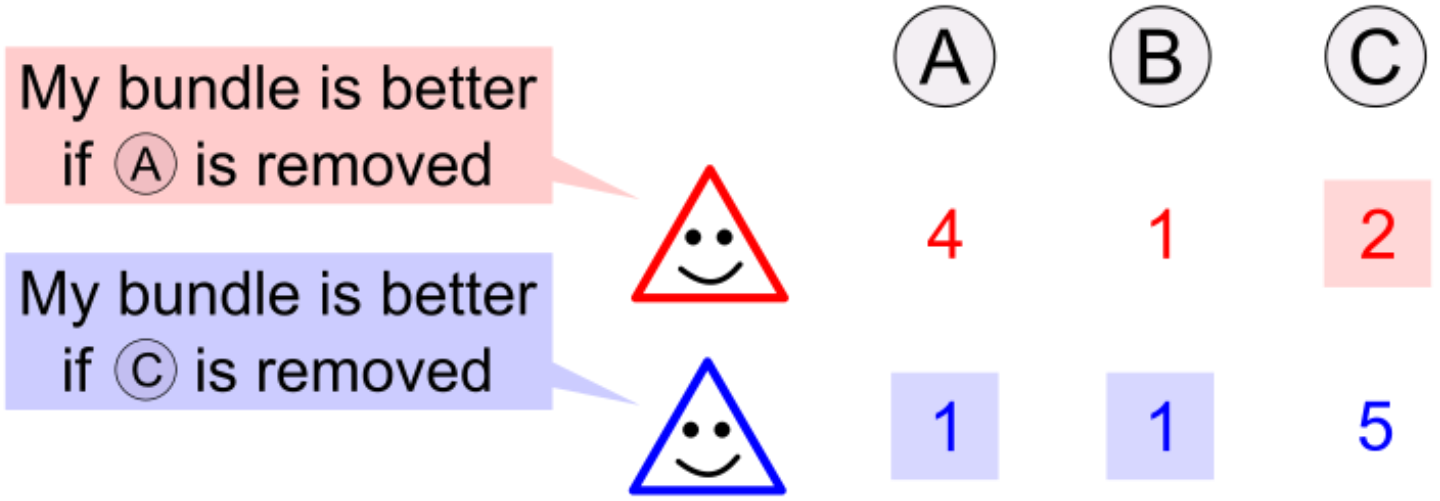
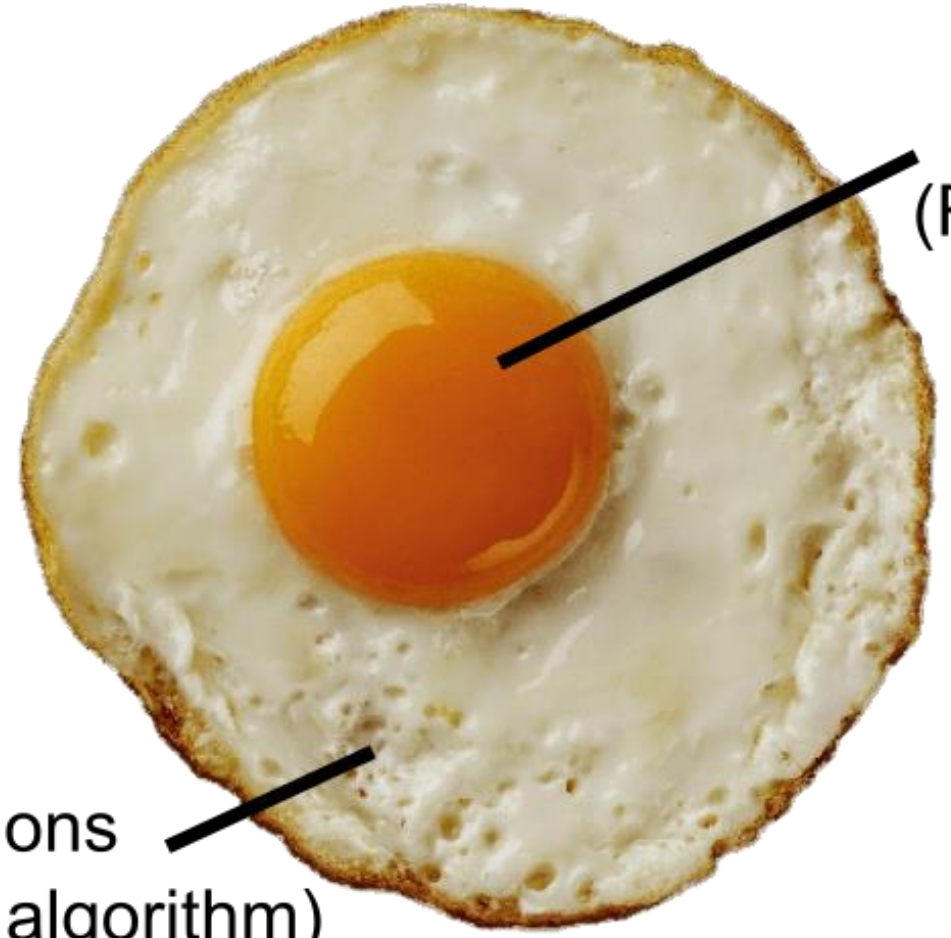


Guaranteed to exist and efficiently computable

Algorithms for finding an EF1 allocation



Additive valuations
(Round-robin algorithm)

Monotone valuations
(Envy-cycle elimination algorithm)

A Limitation of EF1

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I donut think this is fair!

Why are you sad?
Aren't you envy-free up to a car?

A Limitation of EF1

Envy-Freeness Up To Any Good

[Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang, *EC* 2016, *TEAC* 2019]

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	A	B	C	D
	3	3	4	1
	1	1	0	2

Allocation $A = (A_1, \ldots, A_n)$ is EFX if for every pair of agents i, k and for every good $j \in A_k$, we have $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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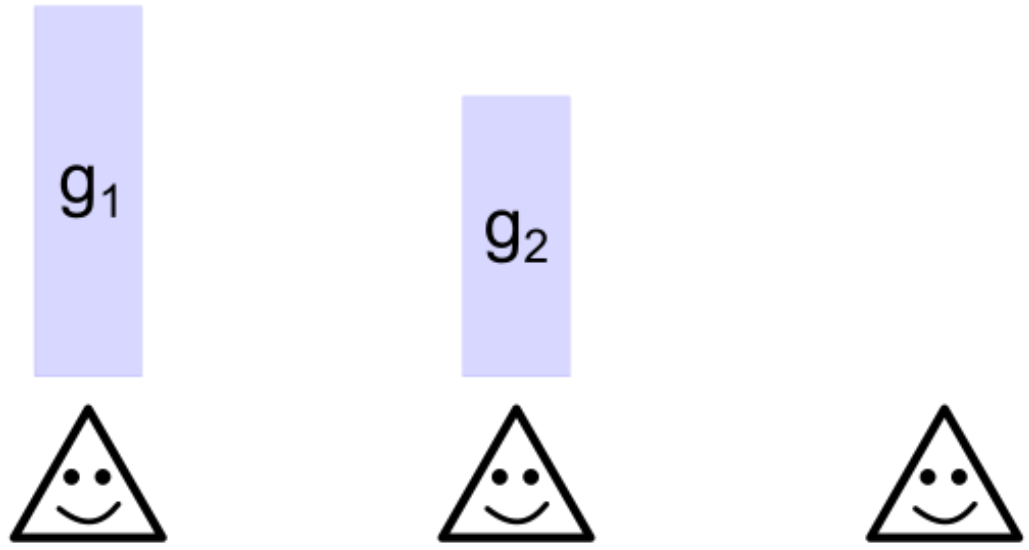

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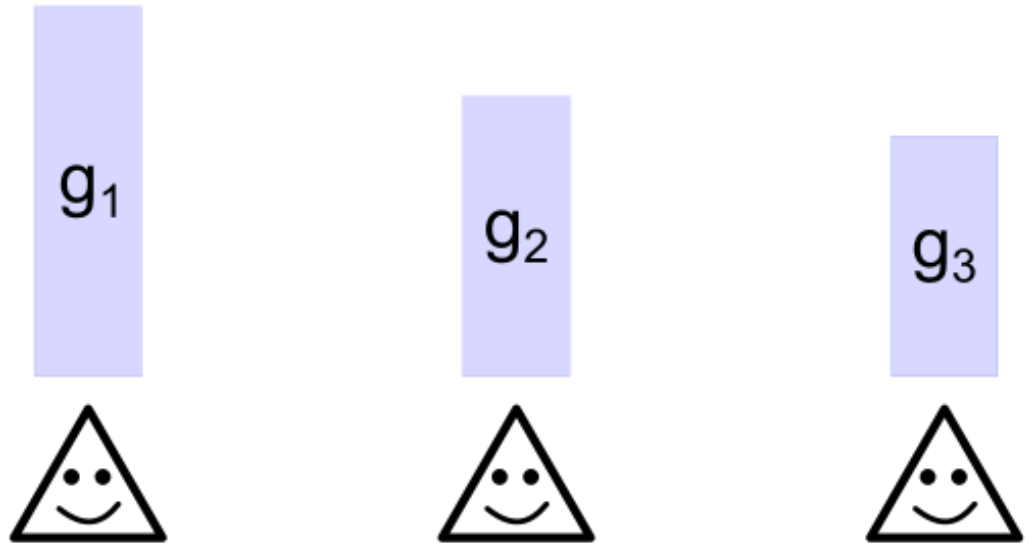

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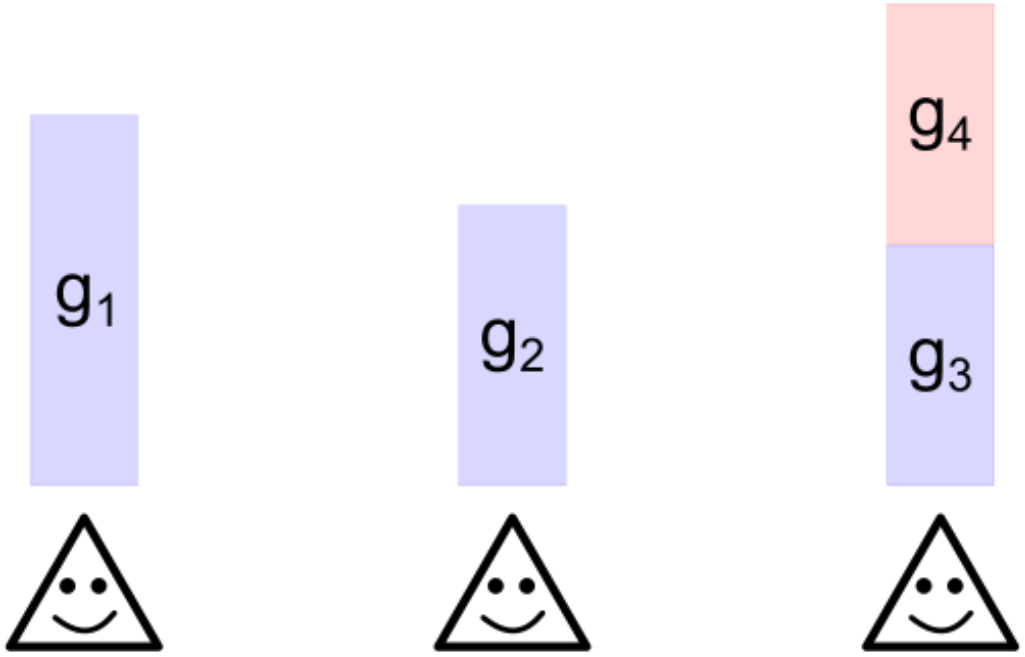

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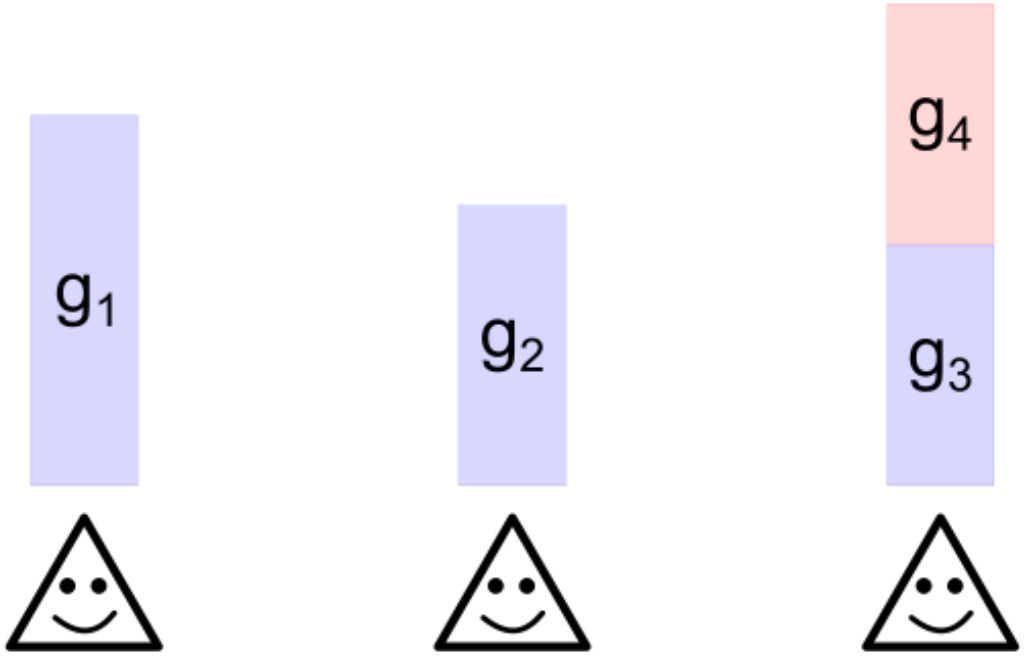

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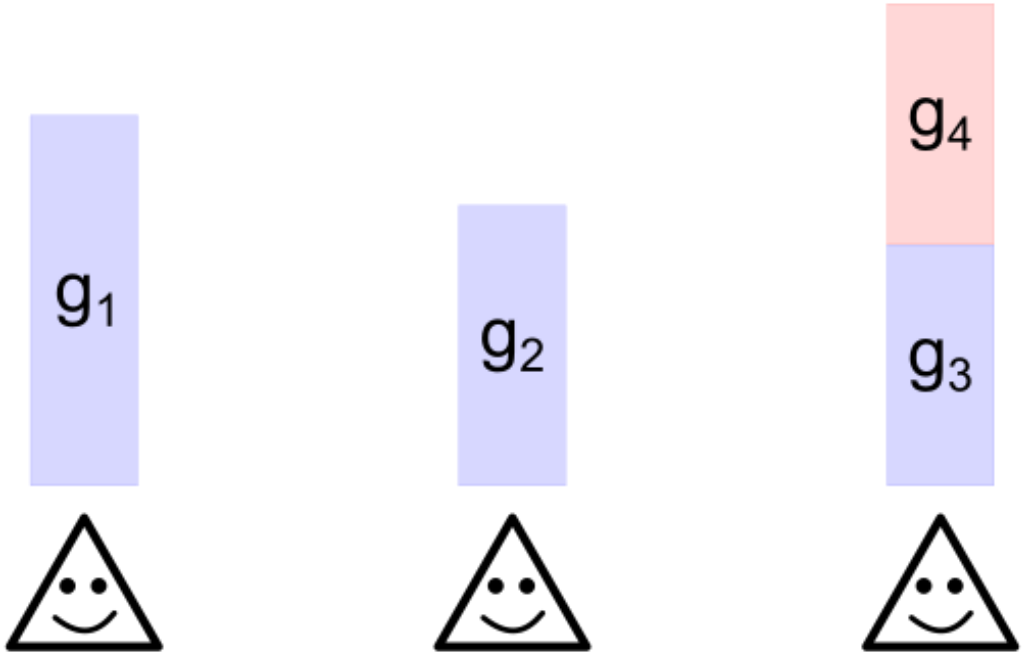

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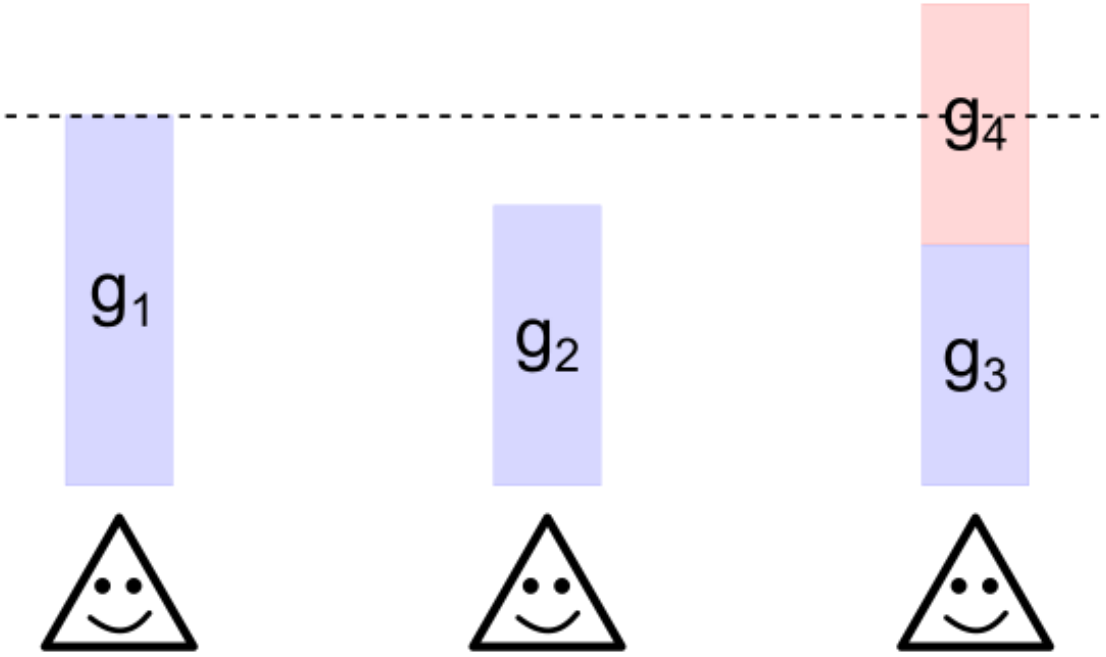

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Most recent good = least valued.

Envy-free up to most recent good.

[Plaut and Roughgarden; *SODA* 2018; *SIDMA* 2020]

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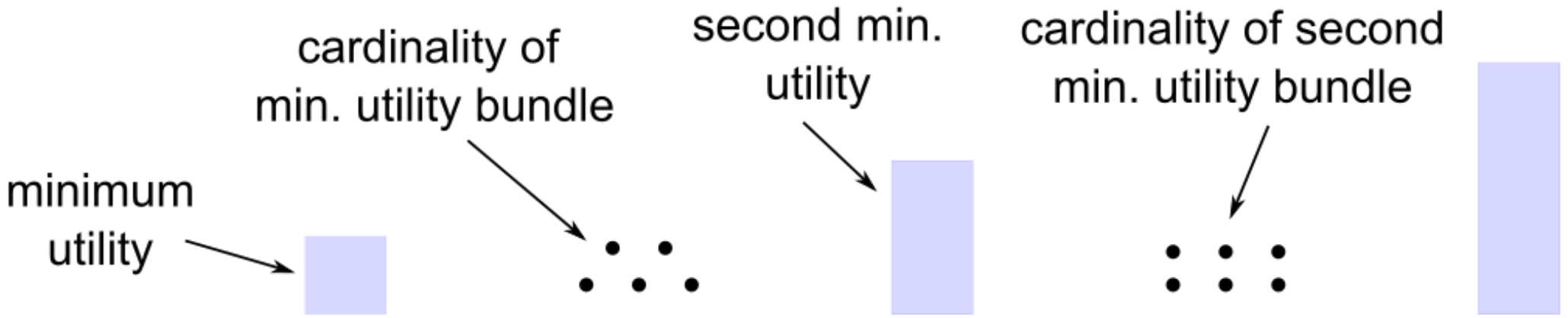

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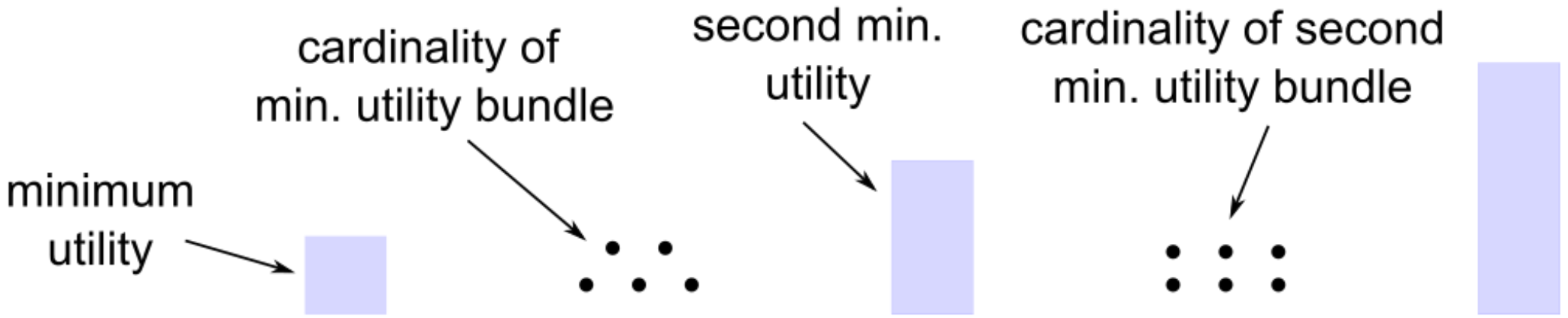

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Finding an EFX allocation can take exponential-in-#goods value queries even for two identical agents with submodular valuations.

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Always exists [Plaut and Roughgarden; *SODA* 2018, *SIDMA* 2020]

identical valuations result + "cut and choose"

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Exists for "2 value" additive instances

[Amanatidis, Birmpas, Filos-Ratsikas, Hollender, and Voudouris, *IJCAI* 2020, *TCS* 2021; Garg and Murhekar, *SAGT* 2021]

Fairness via Charity

EFX-with-charity

[Caragiannis, Gravin, and Huang, *EC* 2019; Chaudhury, Kavitha, Mehlhorn, and Sgouritsa, *SODA* 2020, *SICOMP* 2021]

A partition $(A_1, \ldots, A_n, P)$ into $n+1$ bundles satisfies EFX-with-charity if

- the partial allocation $(A_1, \ldots, A_n)$ is EFX,
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For monotone valuations, an EFX-with-charity allocation always exists.

Minimal Envied Subset

[Chaudhury, Kavitha, Mehlhorn, and Sgouritsa, *SODA* 2020, *SICOMP* 2021]

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$S = \{a,b,c,d\}$

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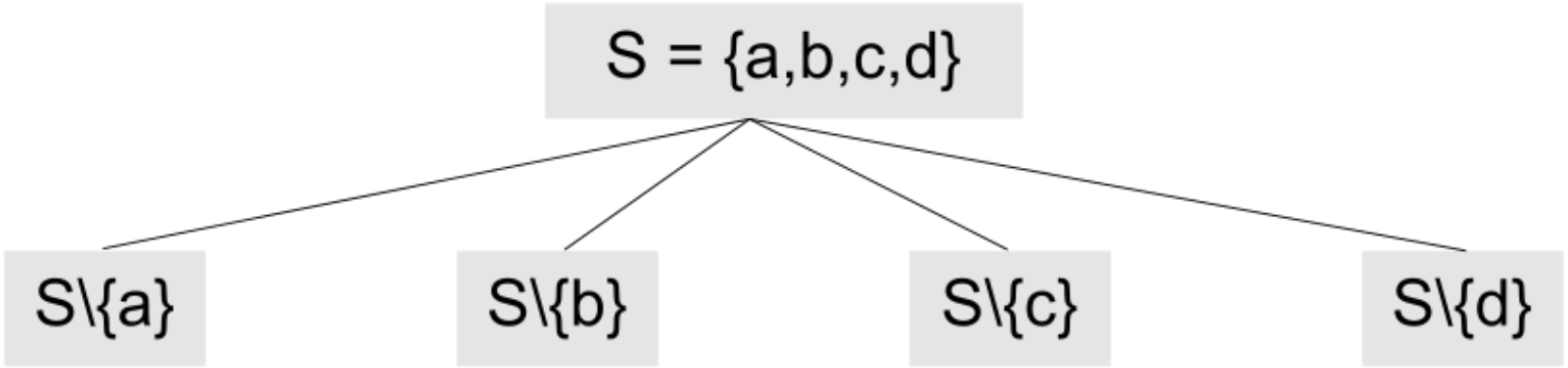

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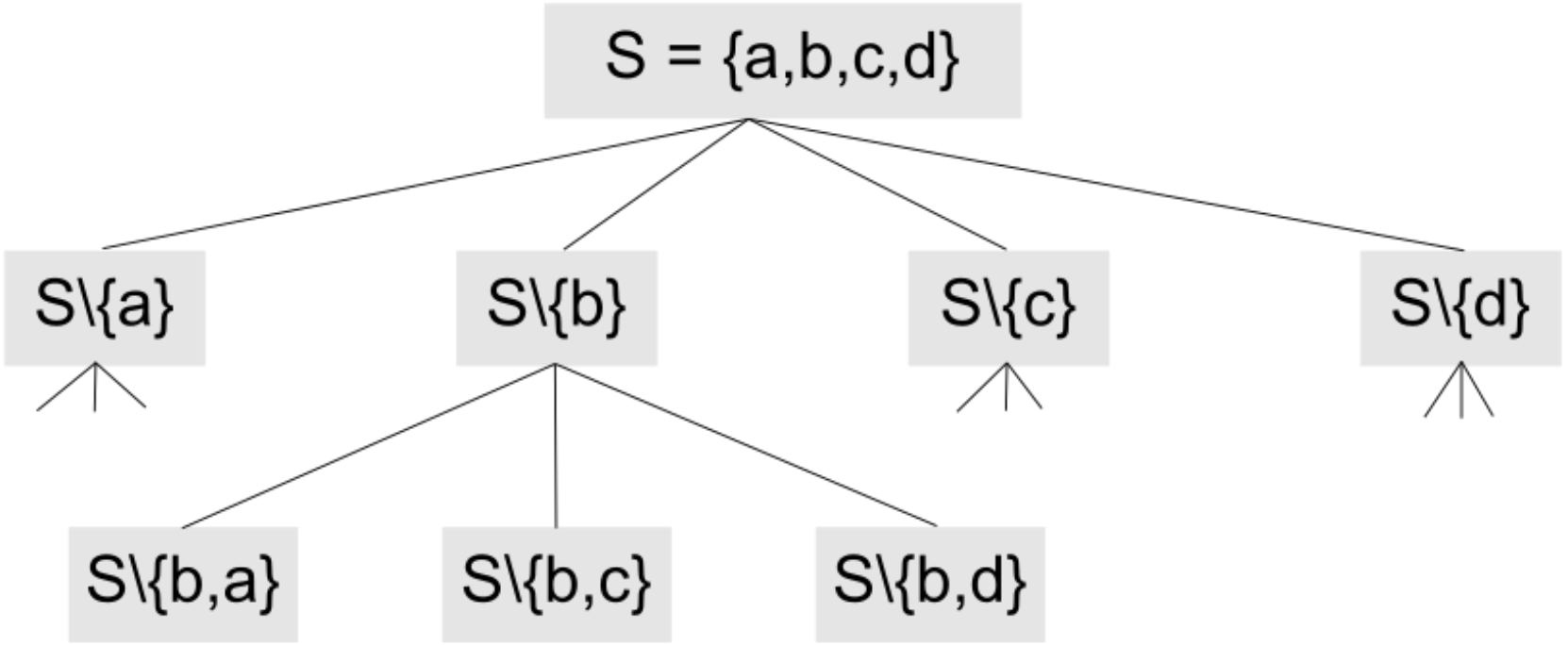

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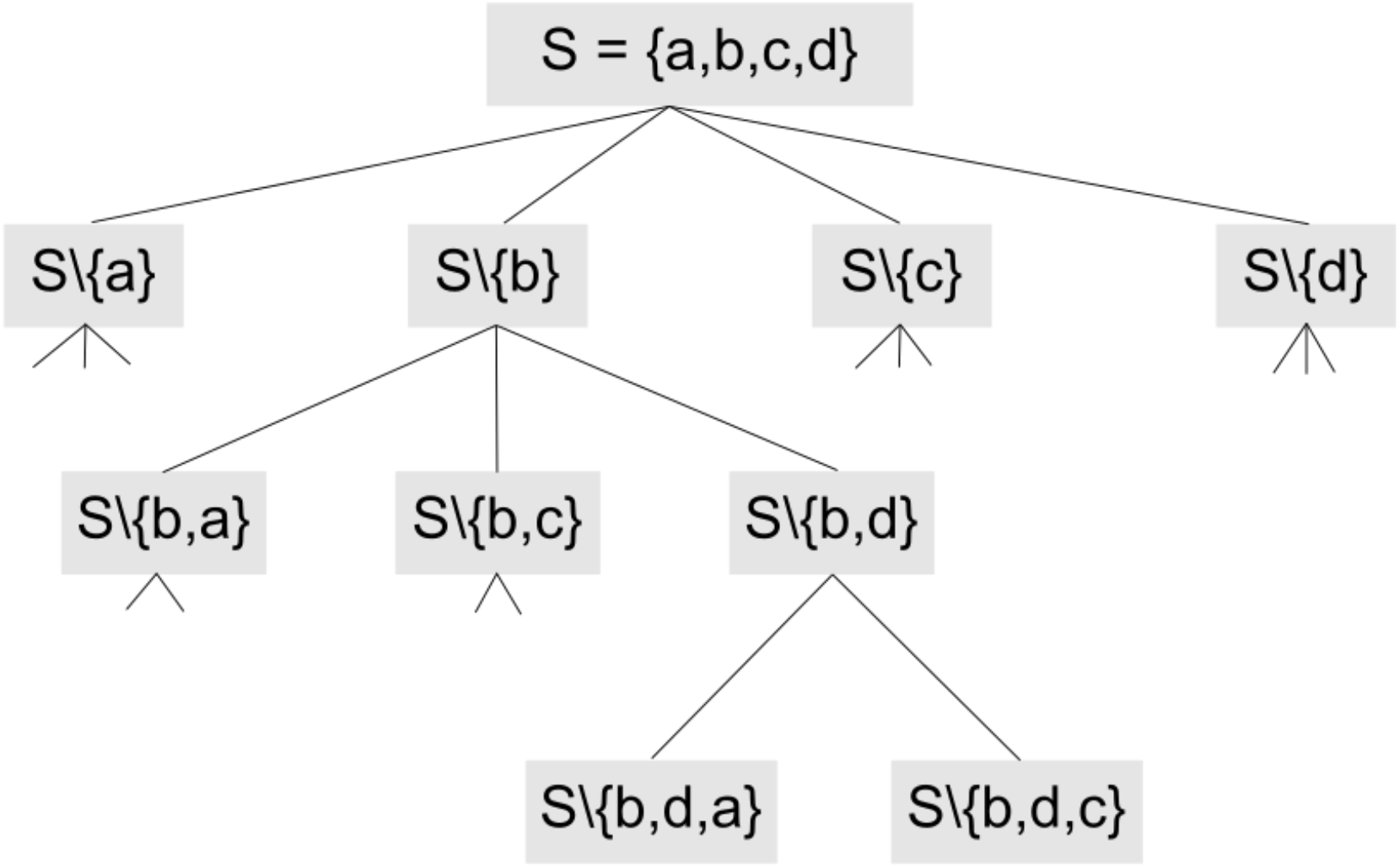

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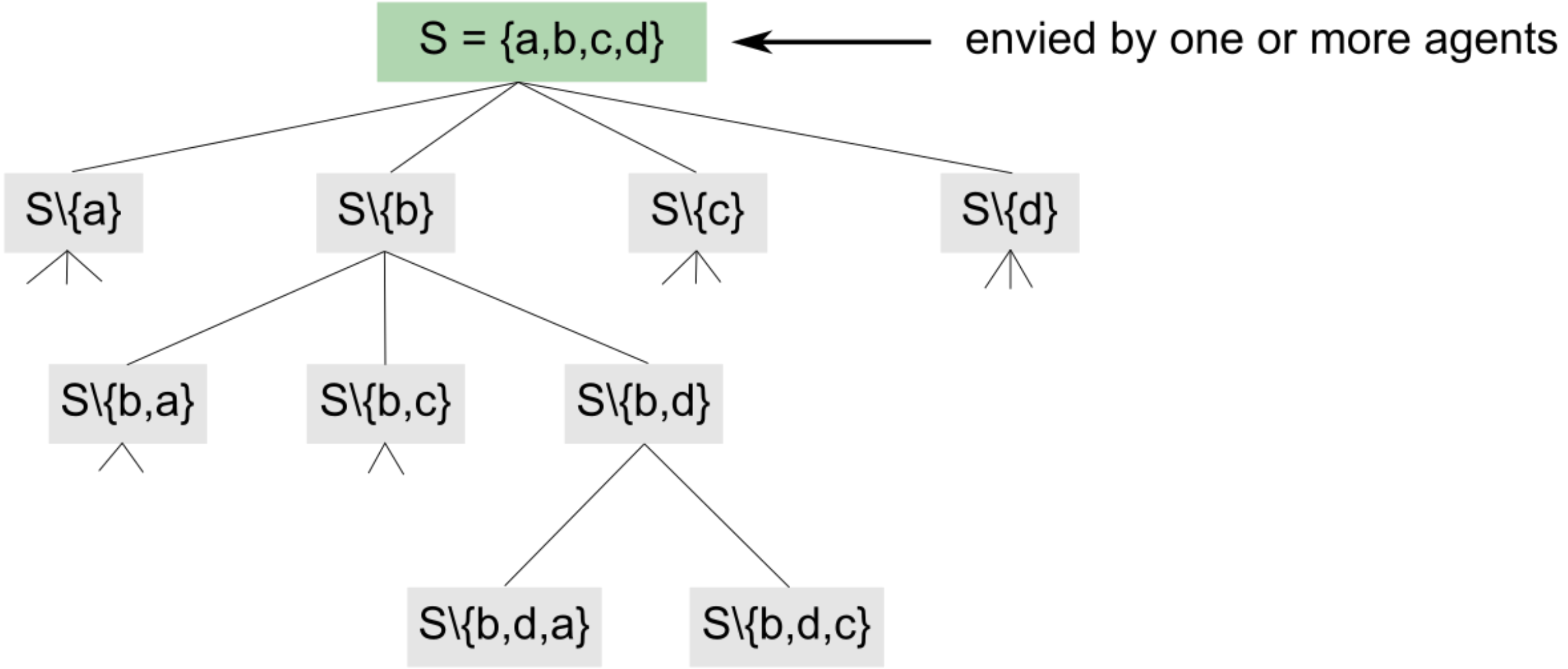

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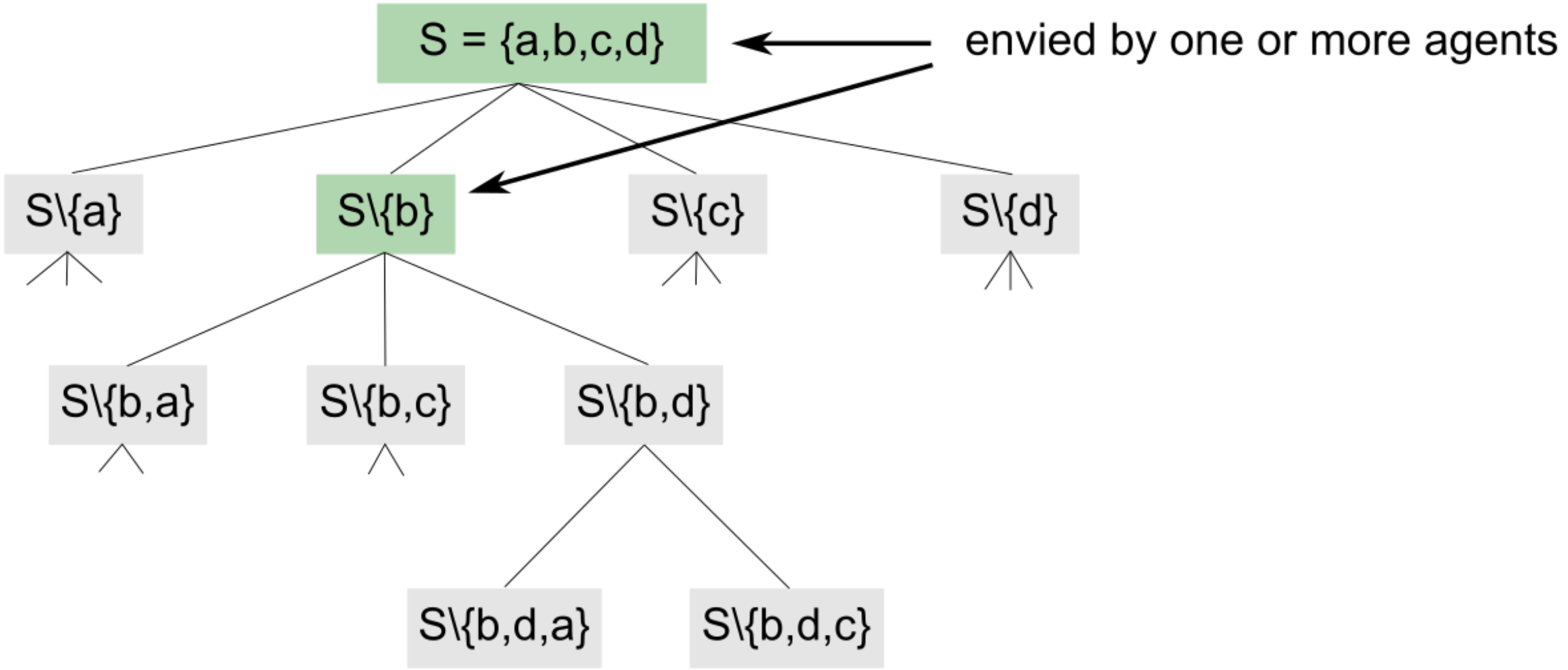

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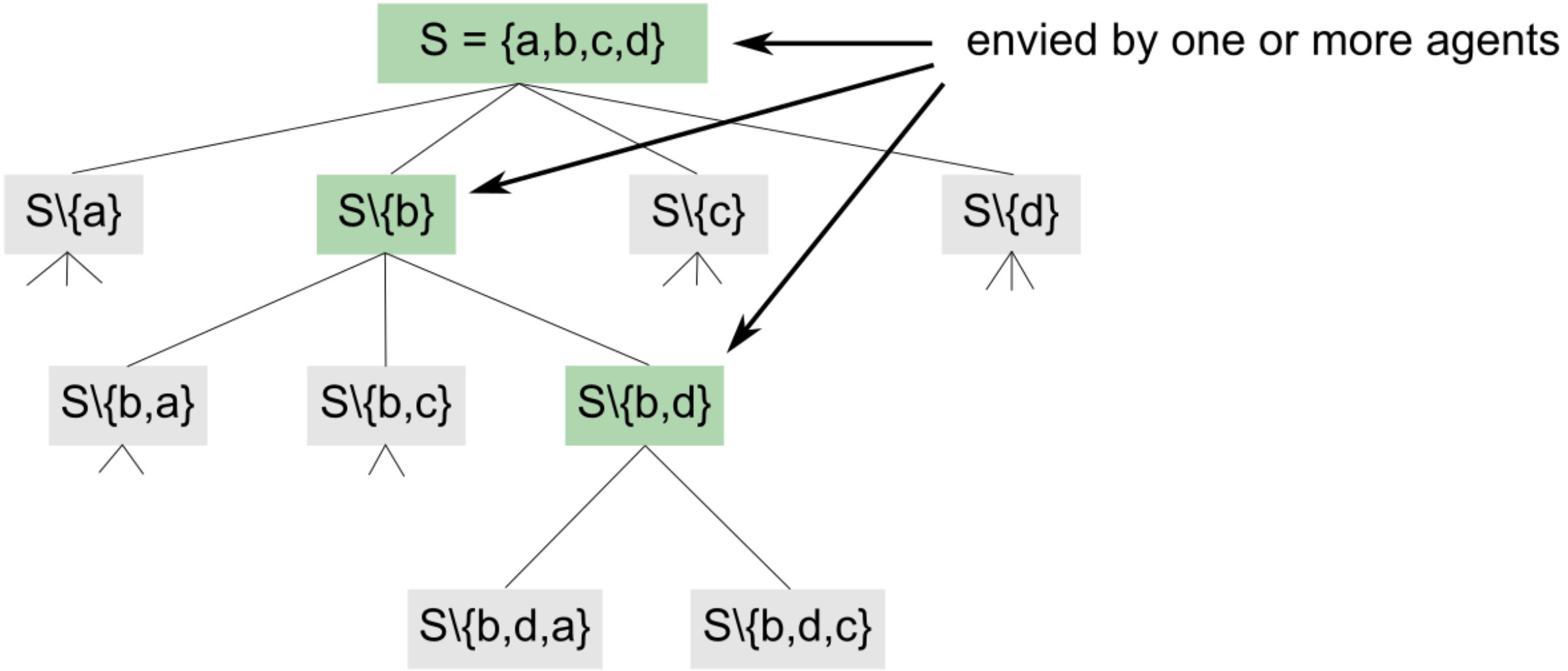

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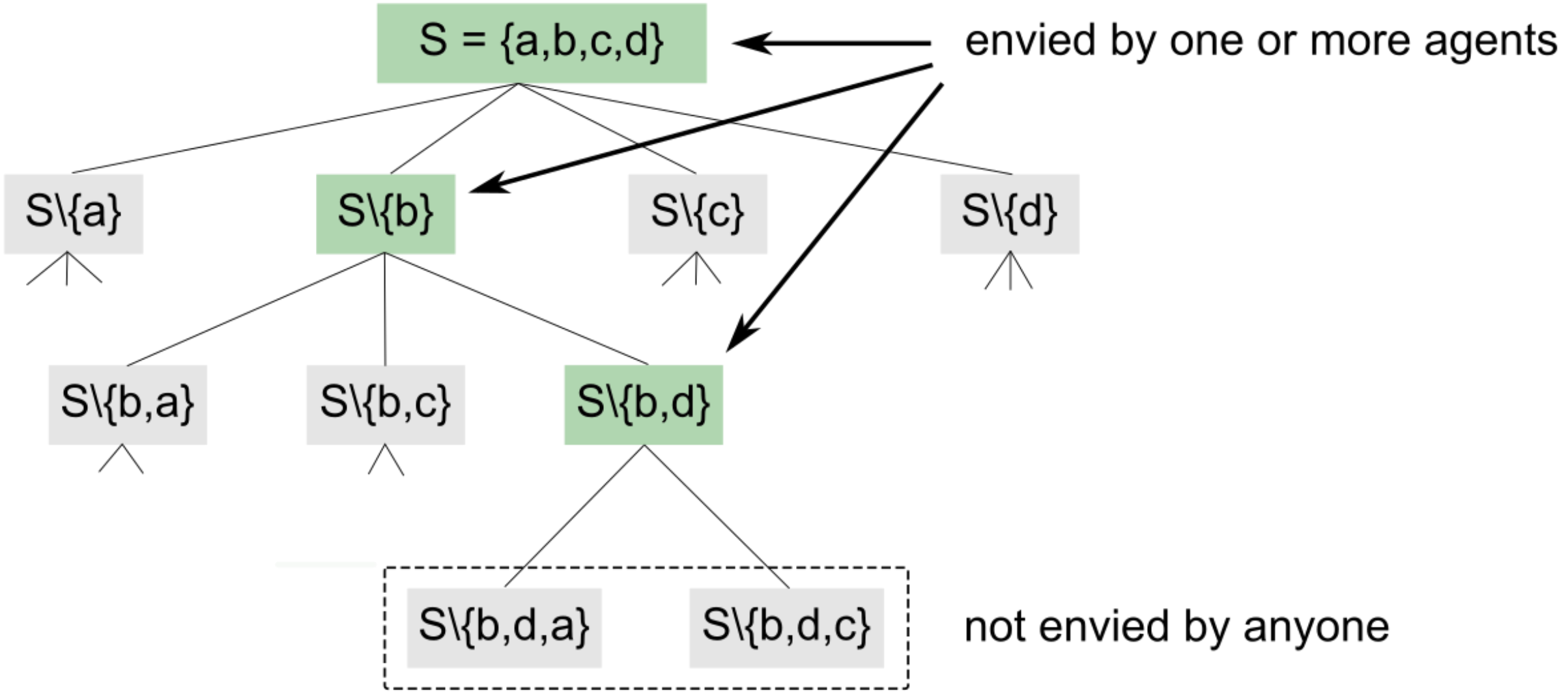

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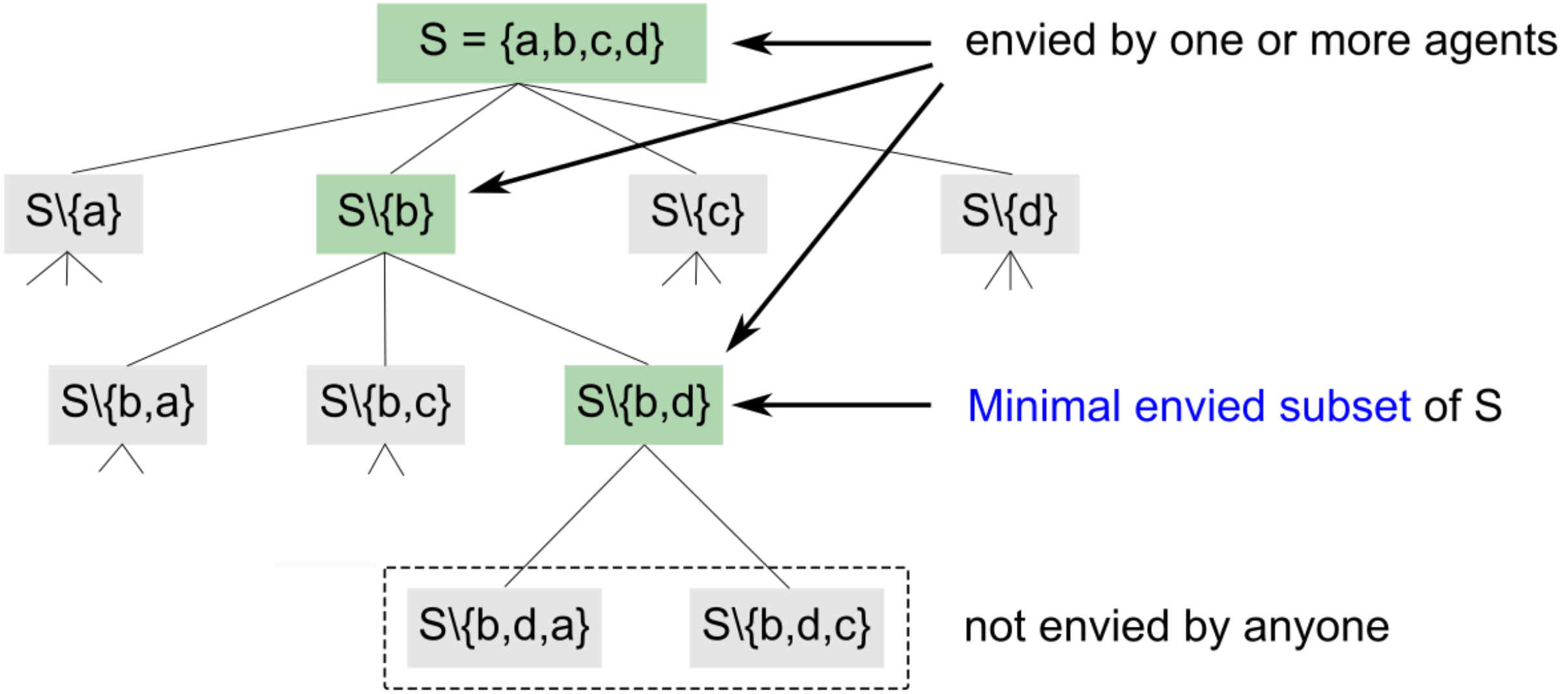

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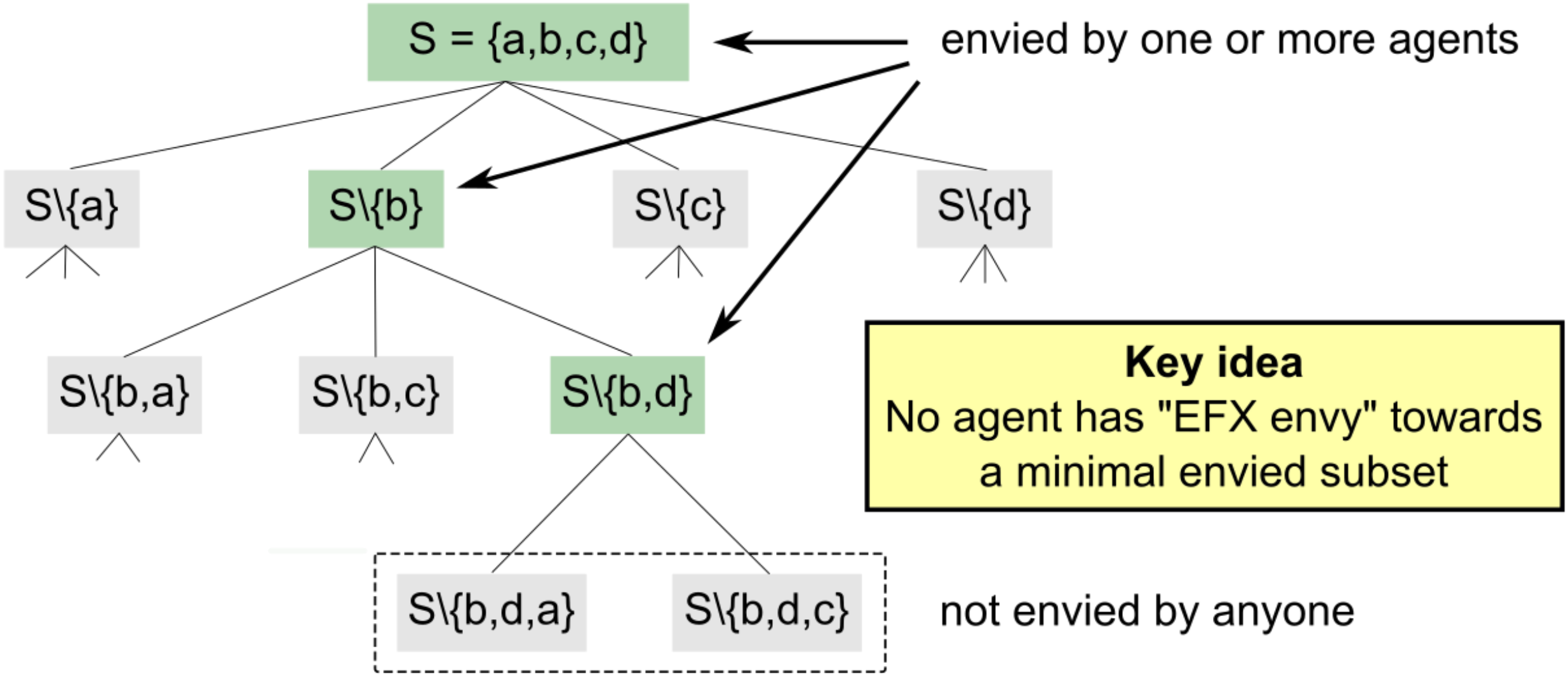

Key idea

No agent has "EFX envy" towards a minimal envied subset

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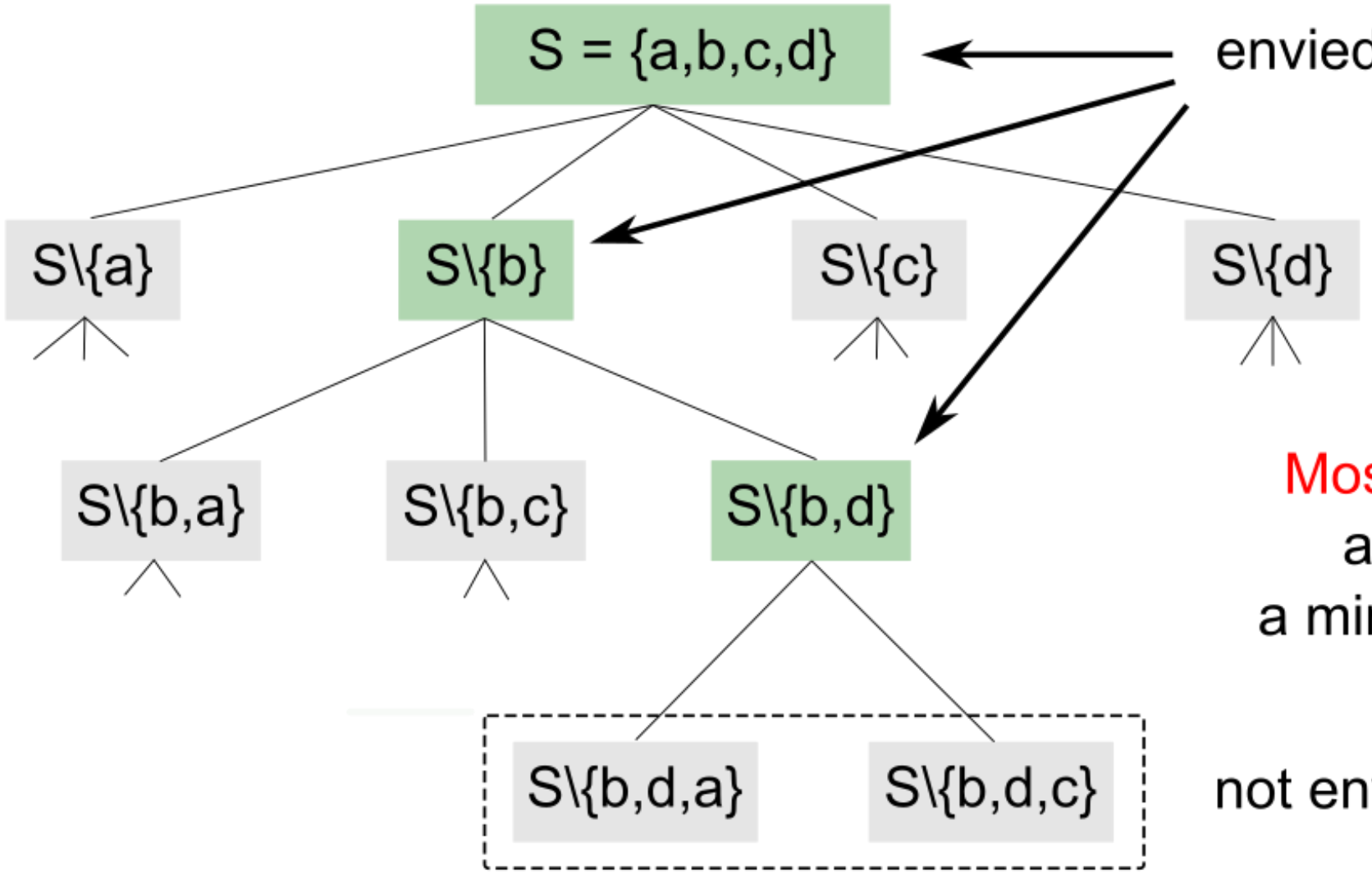

envied by one or more agents

Most envious agent of S =
any agent that envies
a minimal envied subset of S

not envied by anyone

Achieving EFX-with-charity

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make progress towards these

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Start with everything unallocated (i.e., all goods in the pool P).

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- Rule 2
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Otherwise, return the current allocation.

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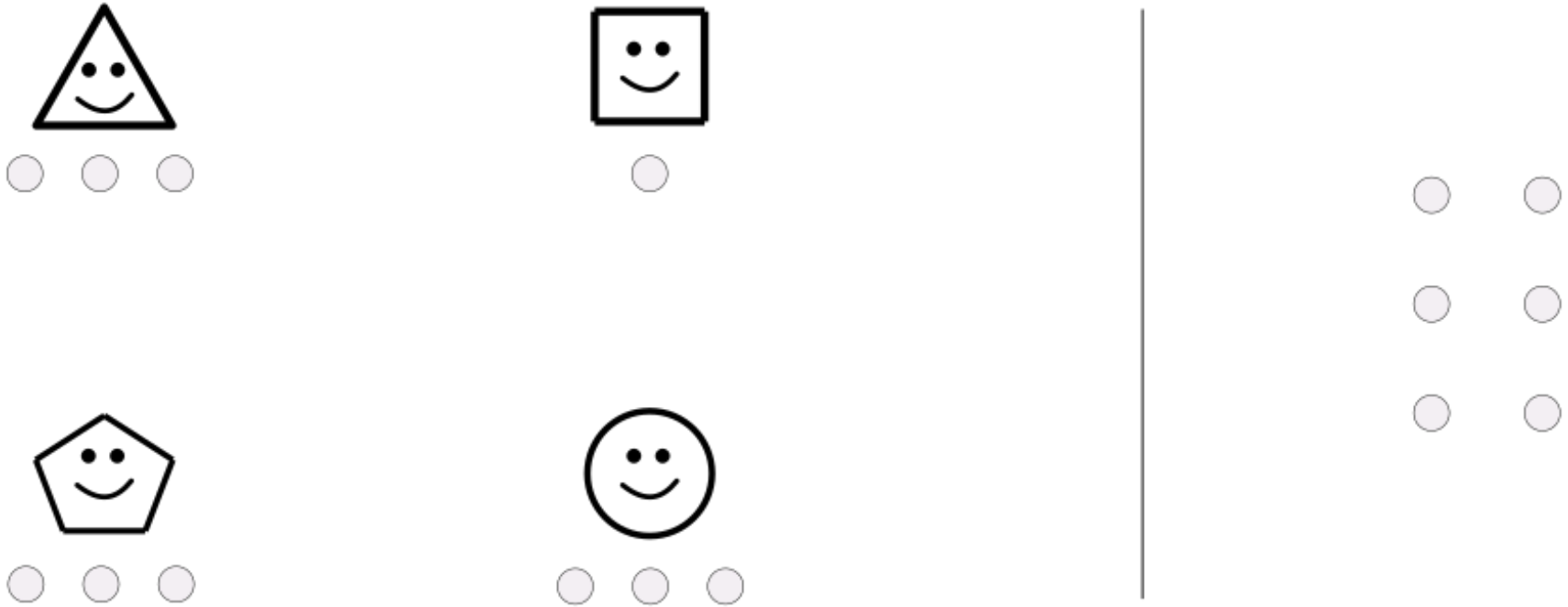

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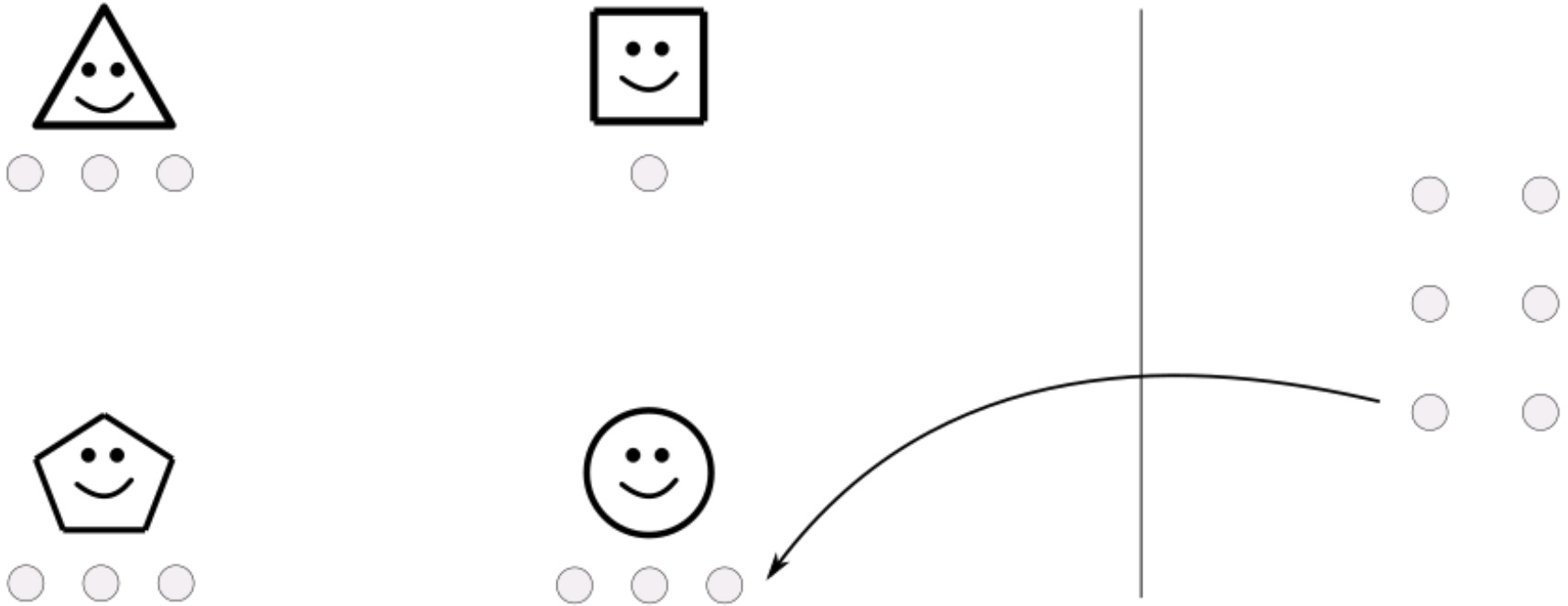

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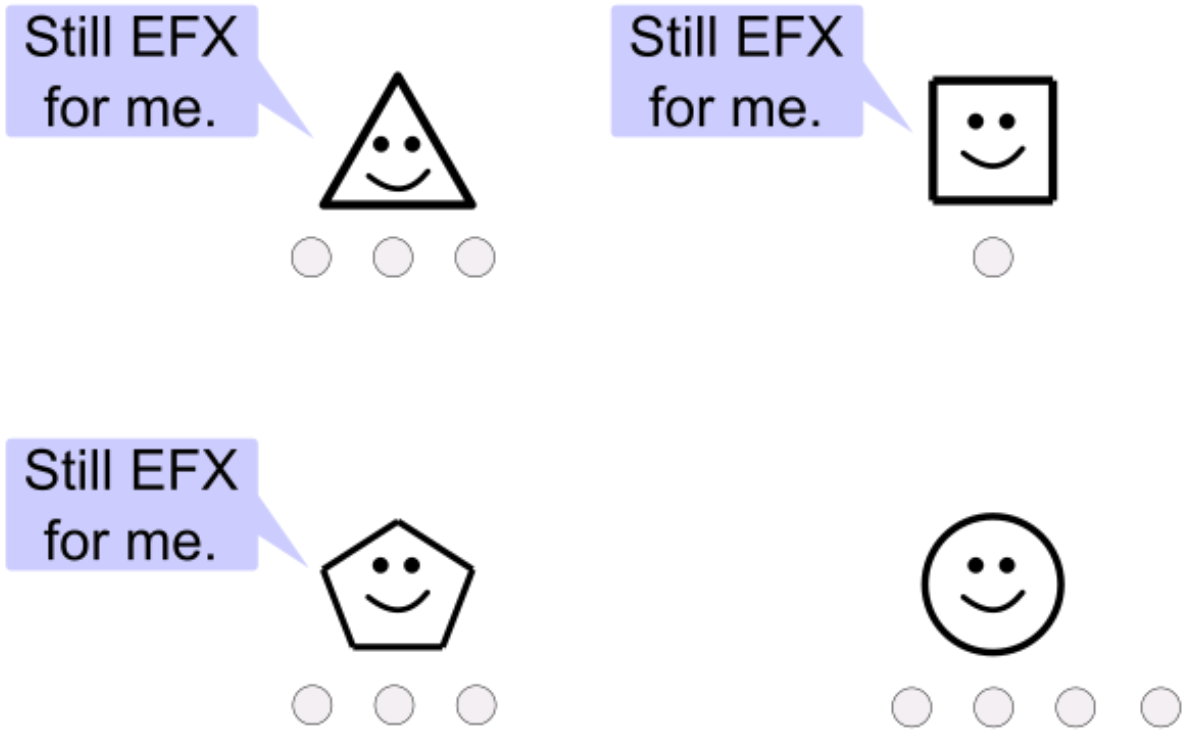

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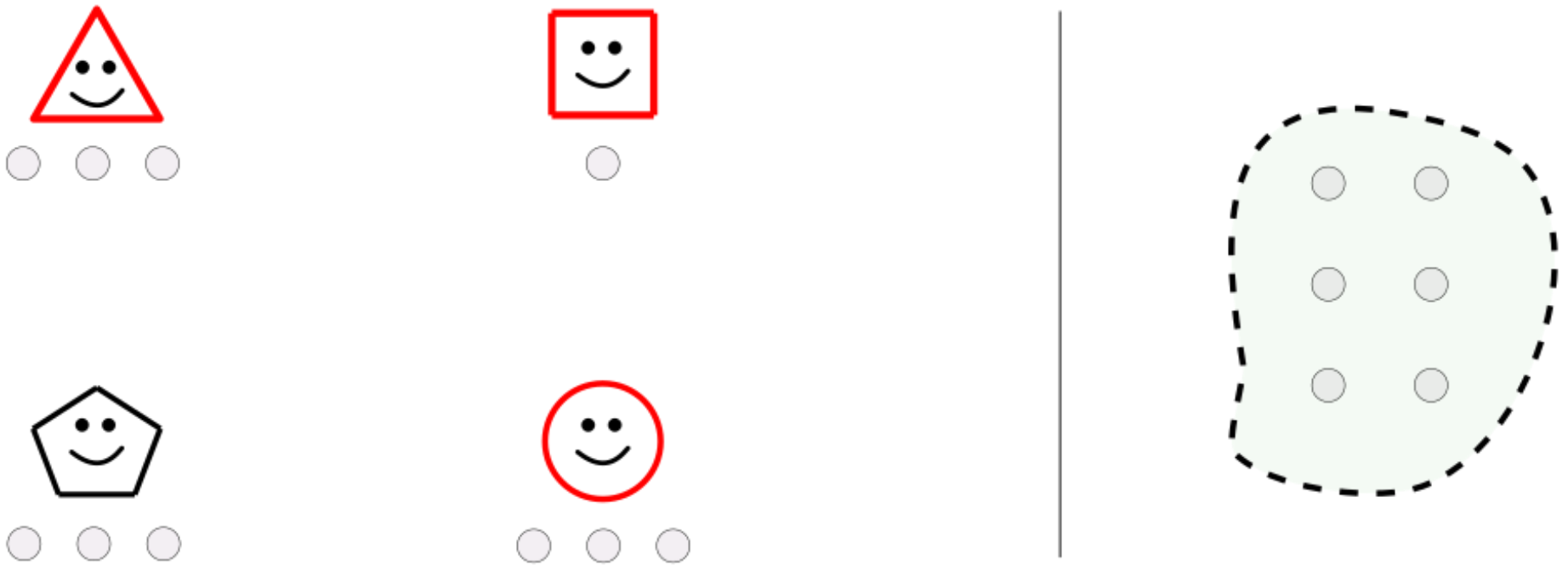

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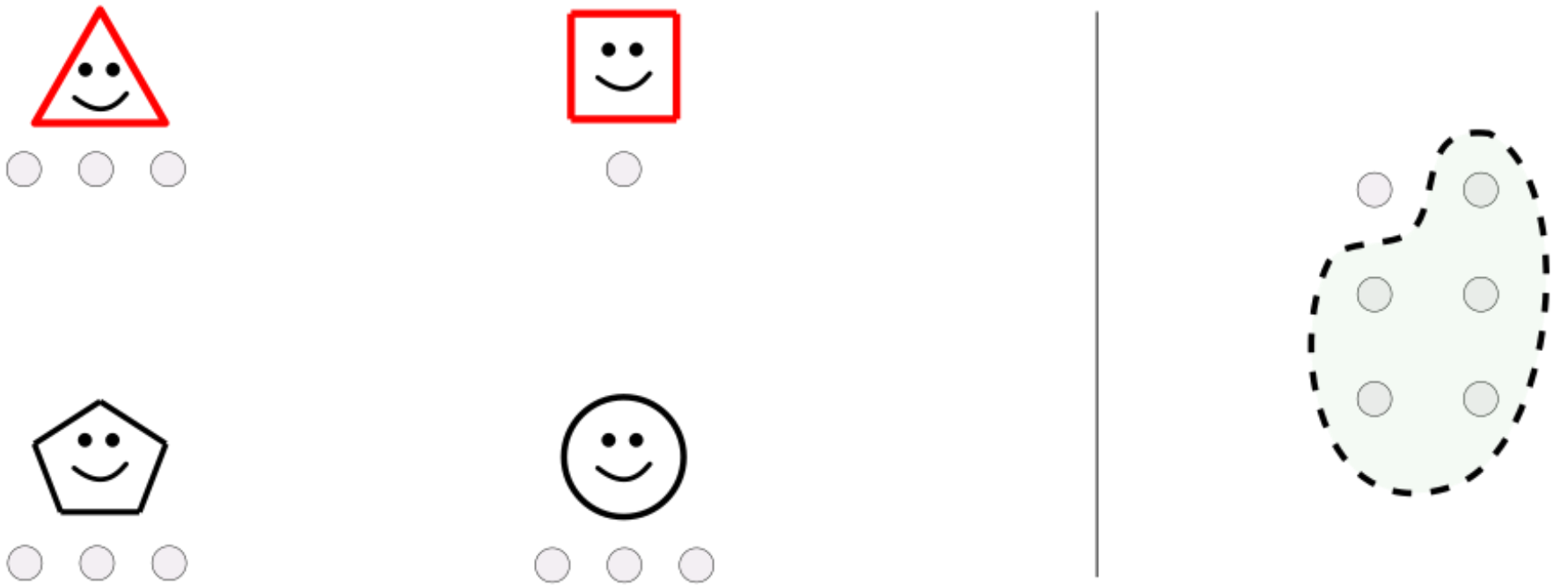

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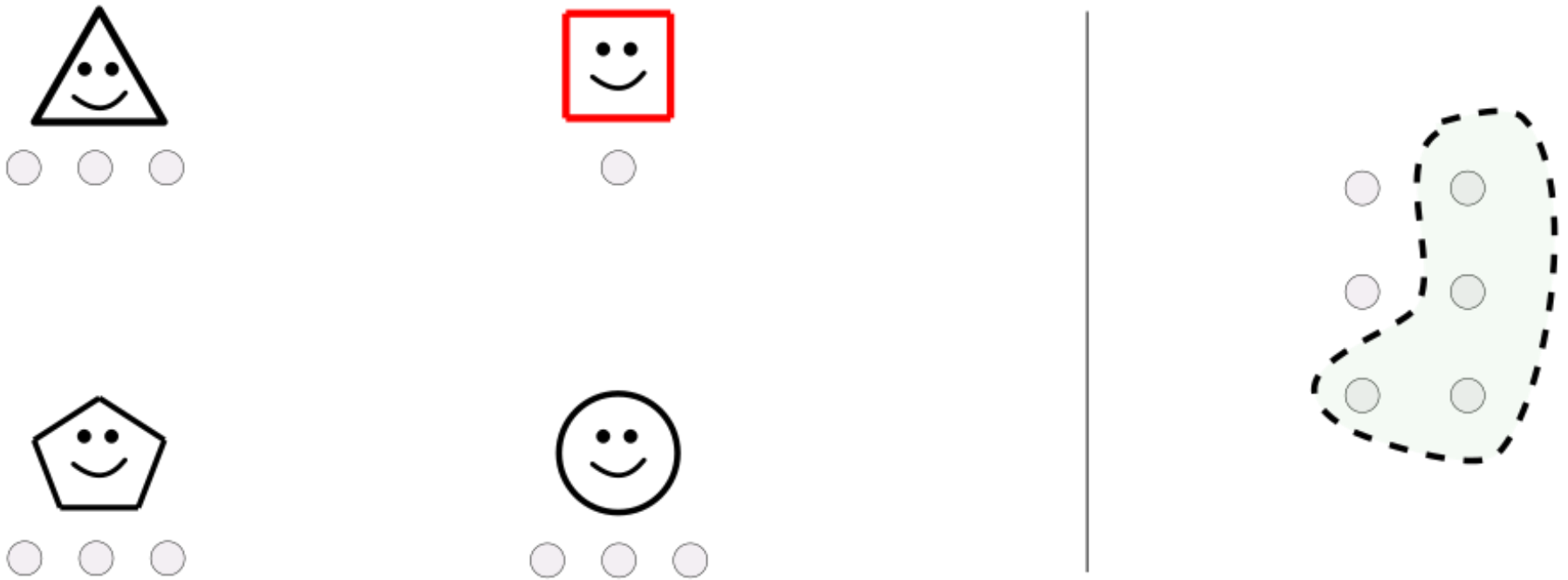

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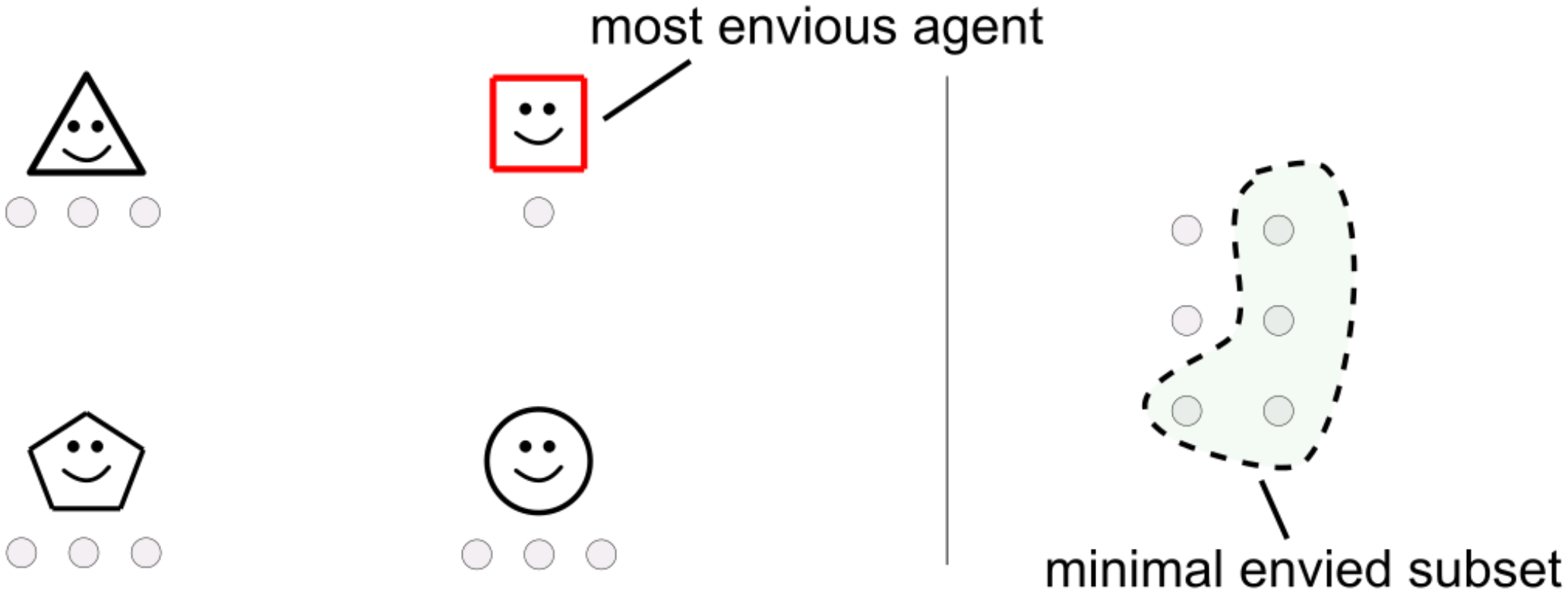

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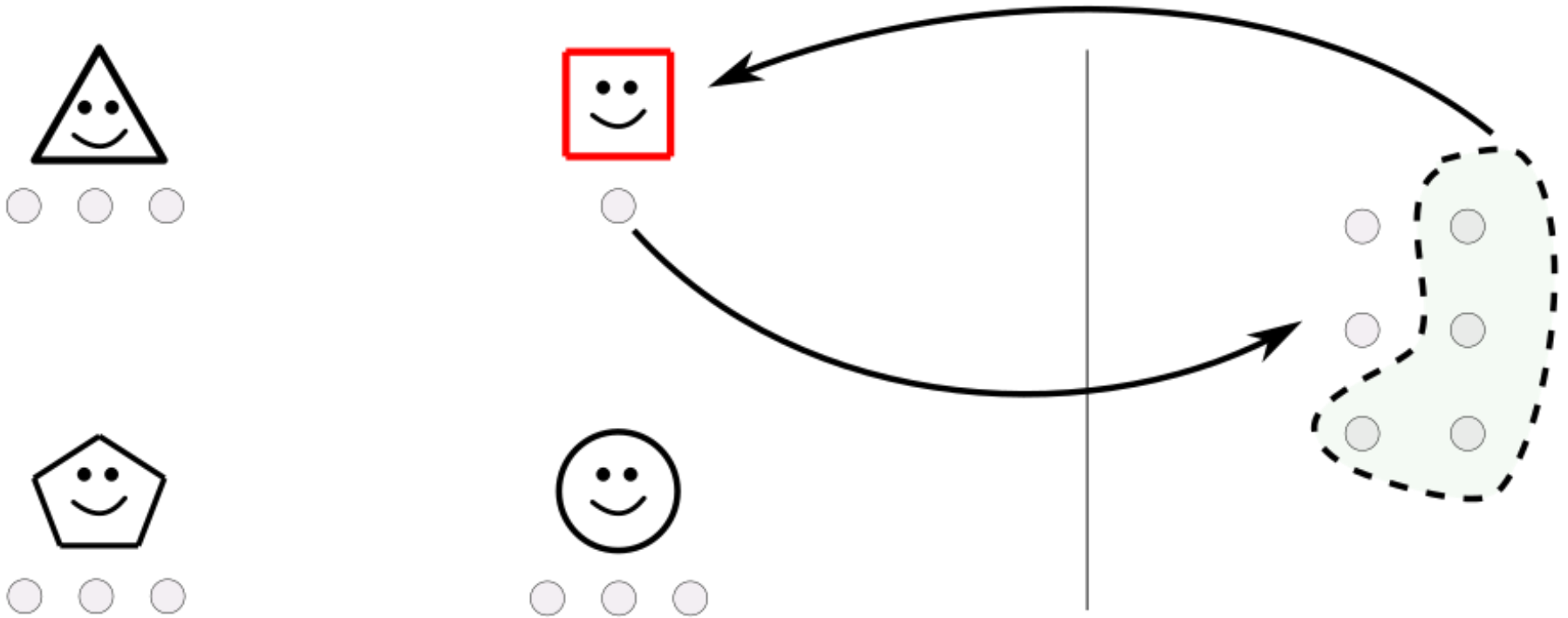

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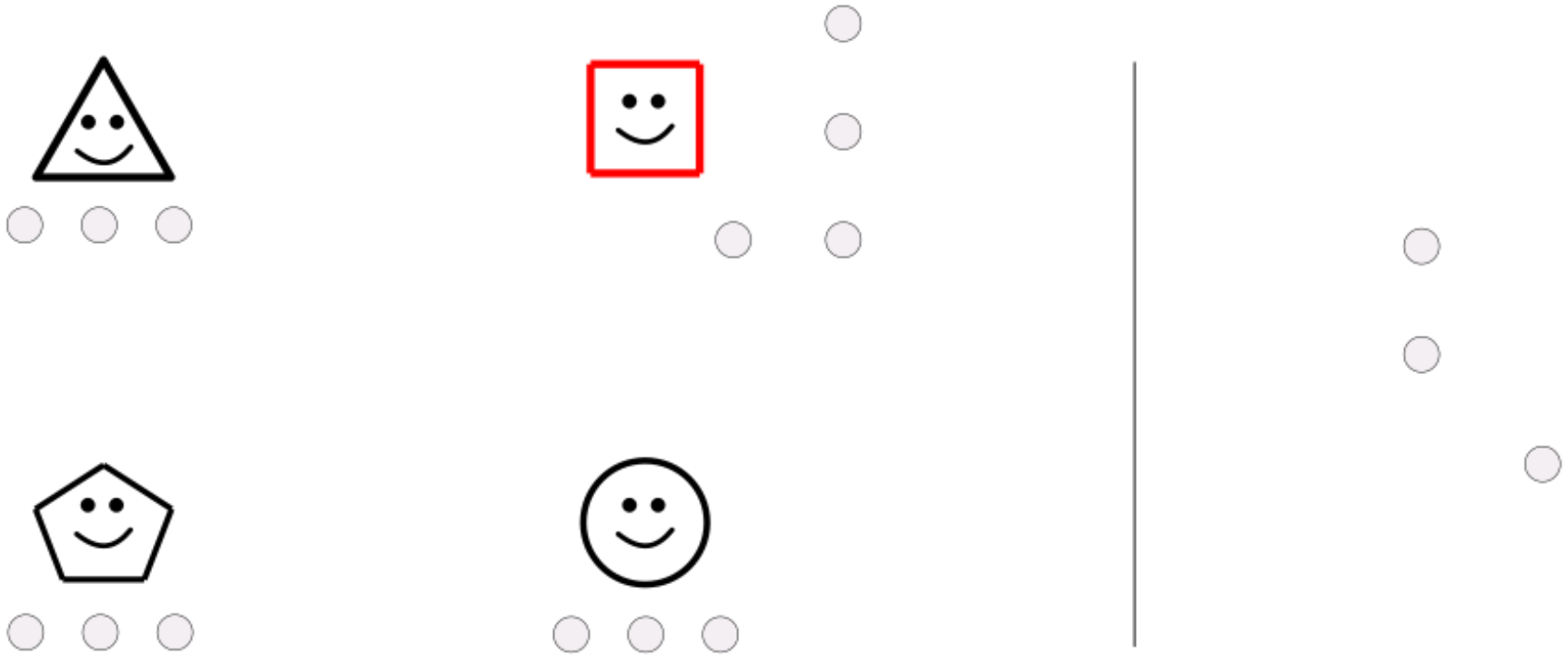

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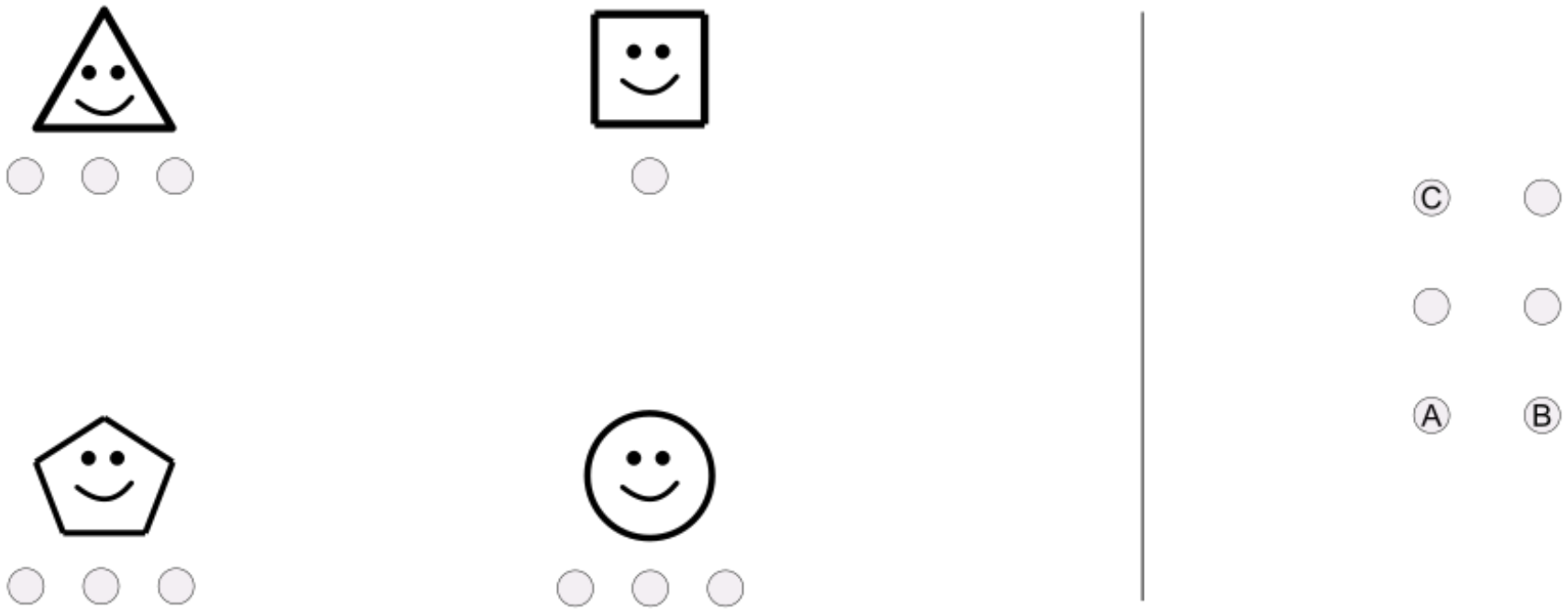

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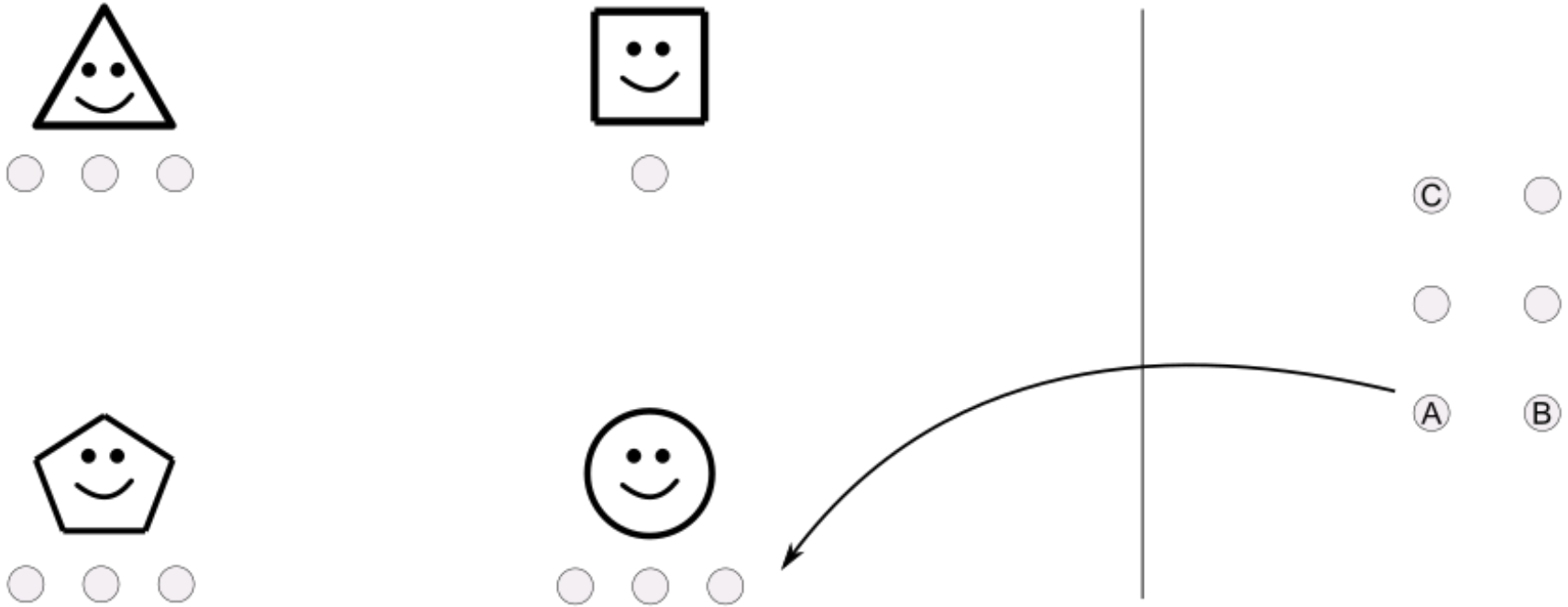

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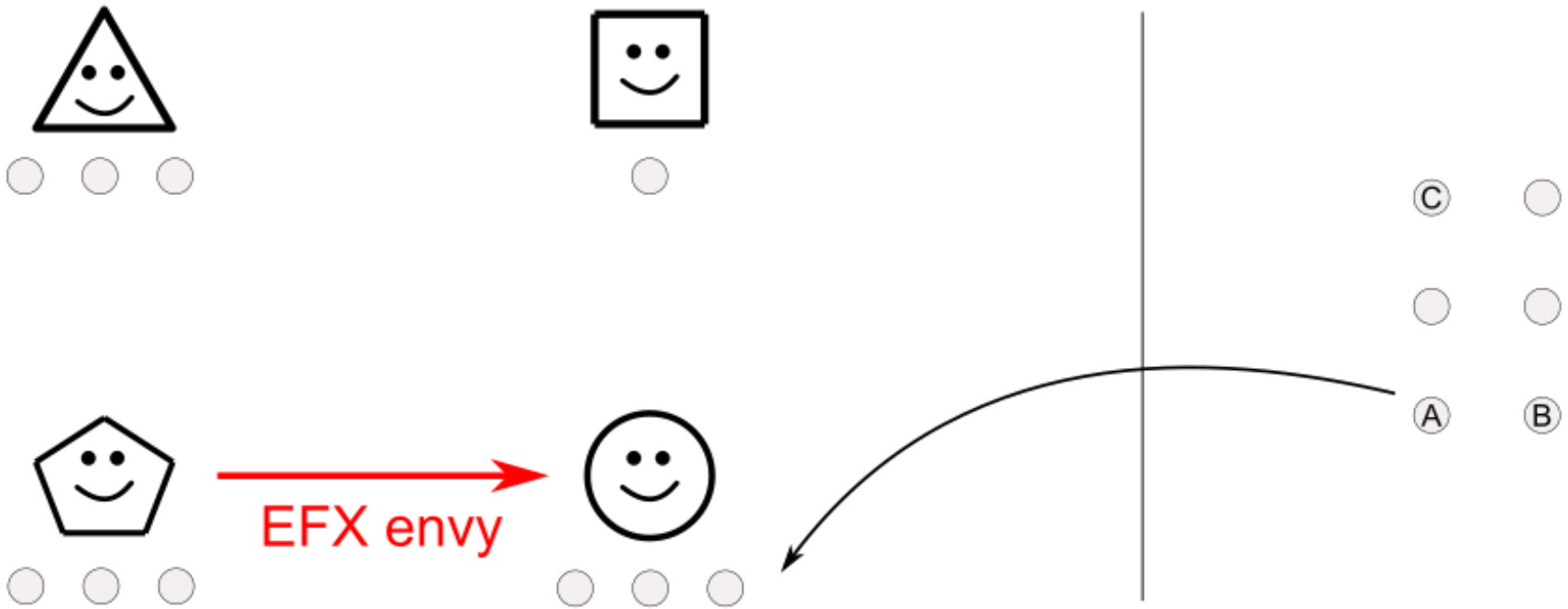

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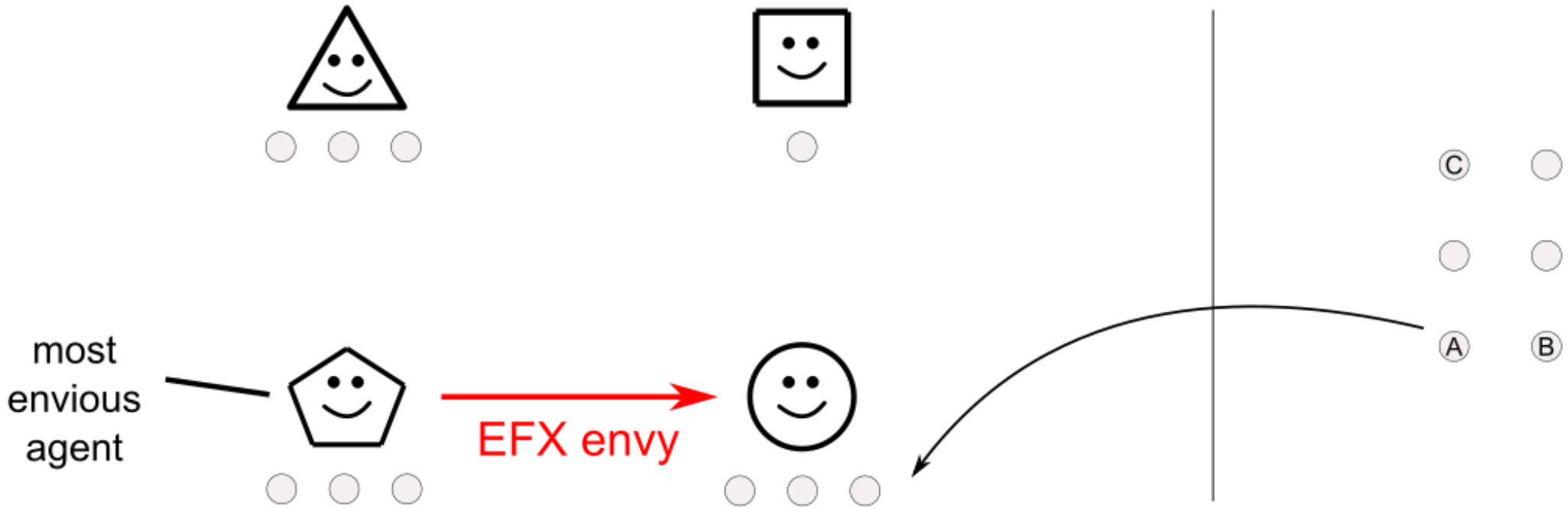

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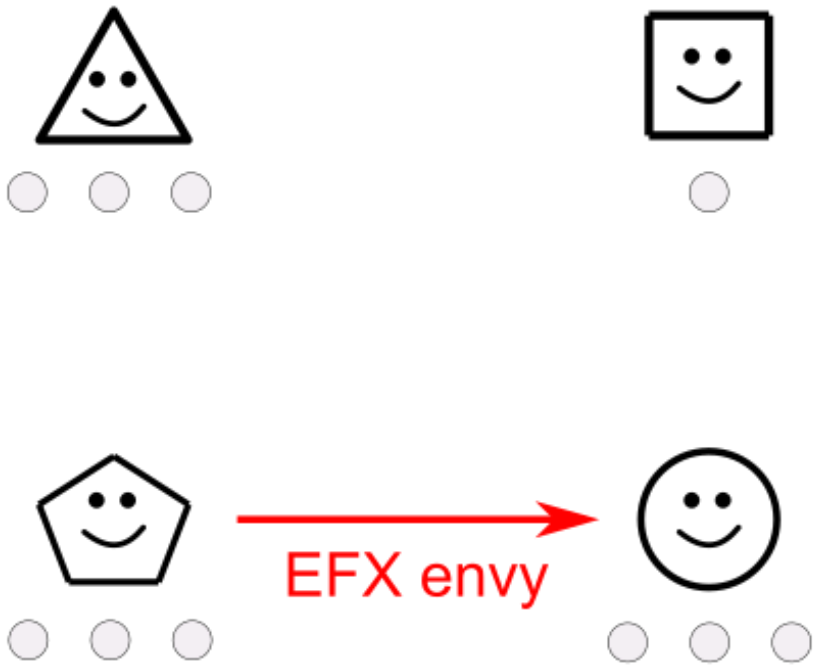

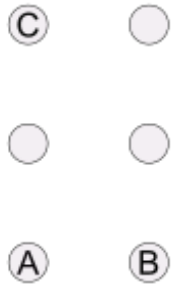

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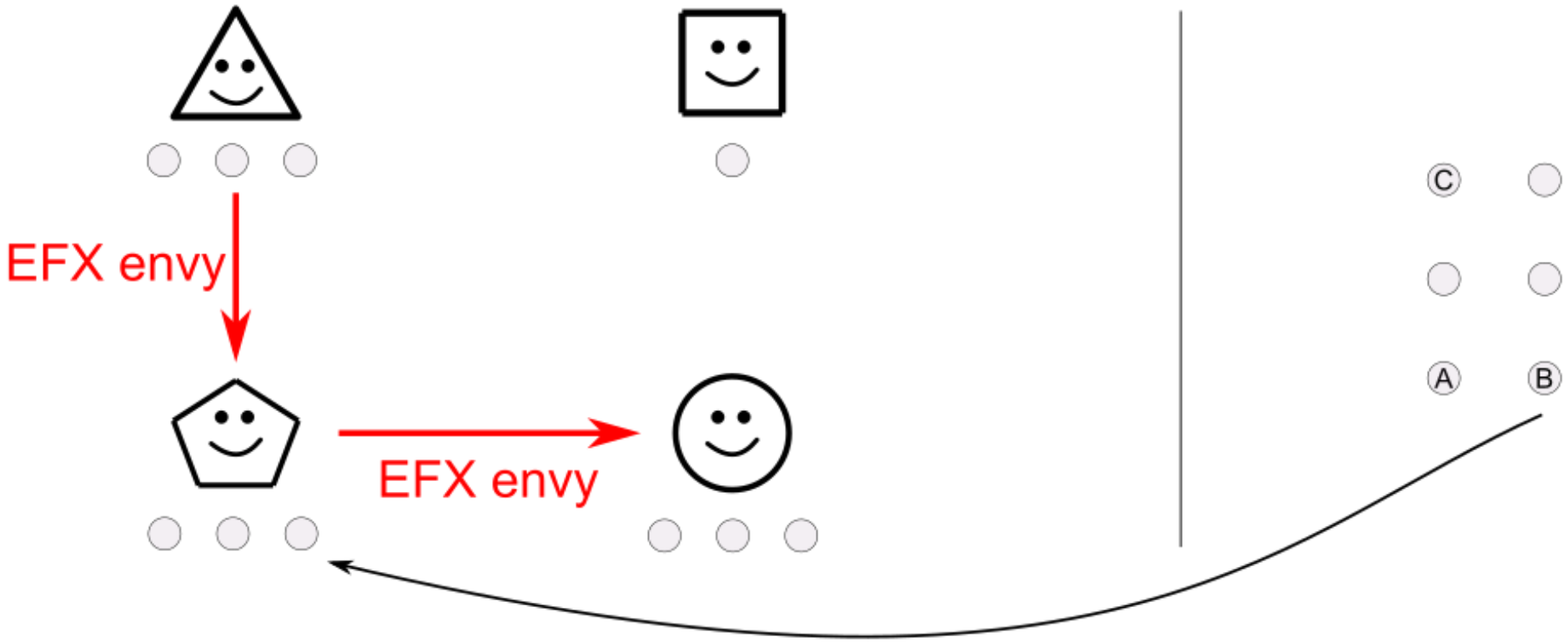

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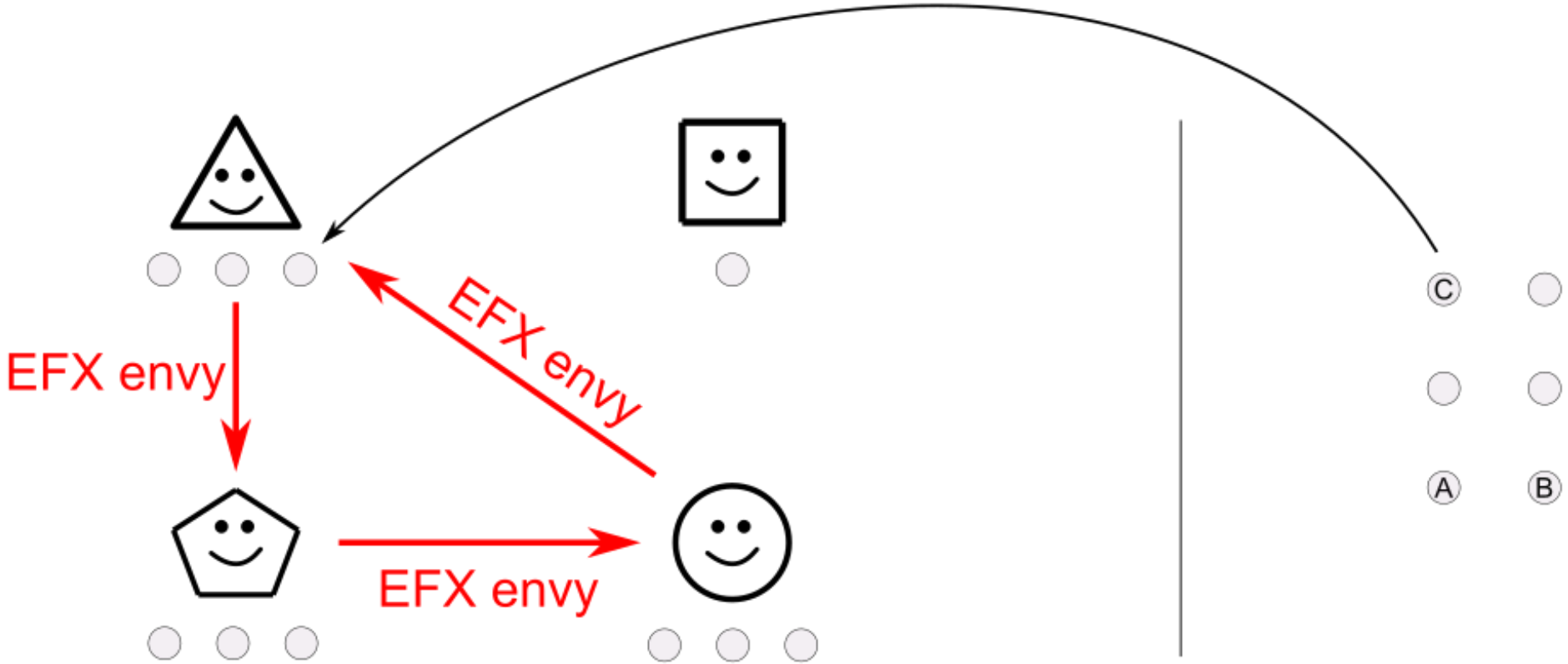

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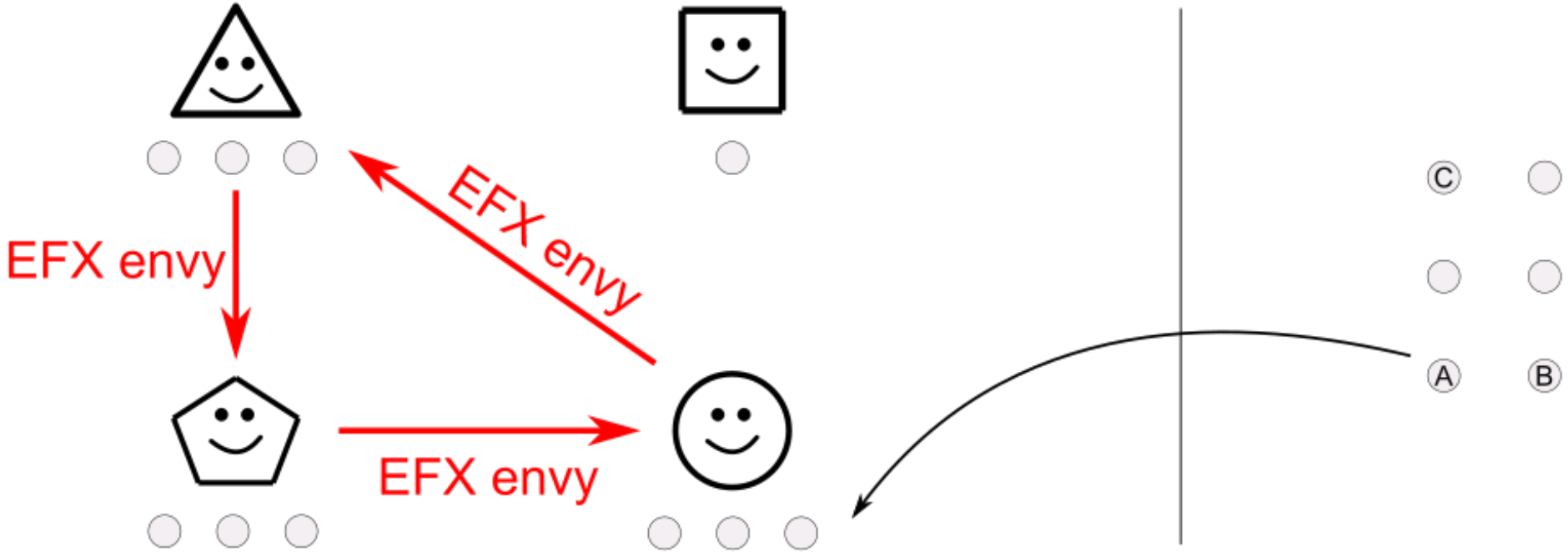

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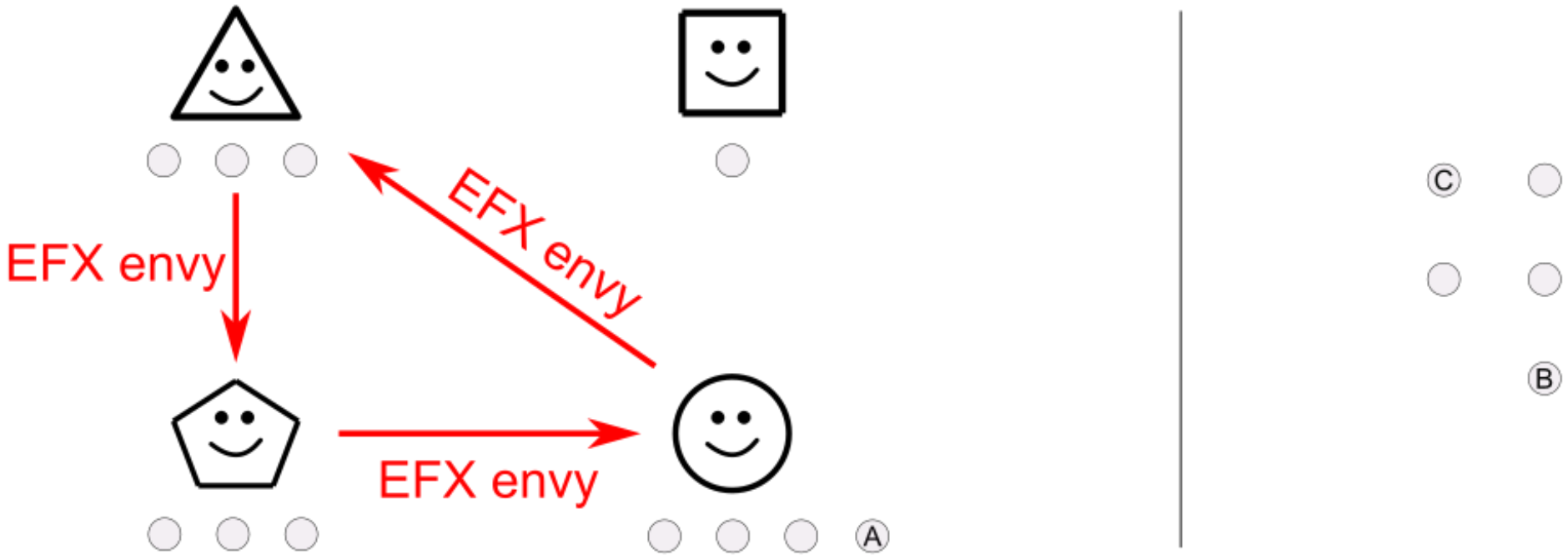

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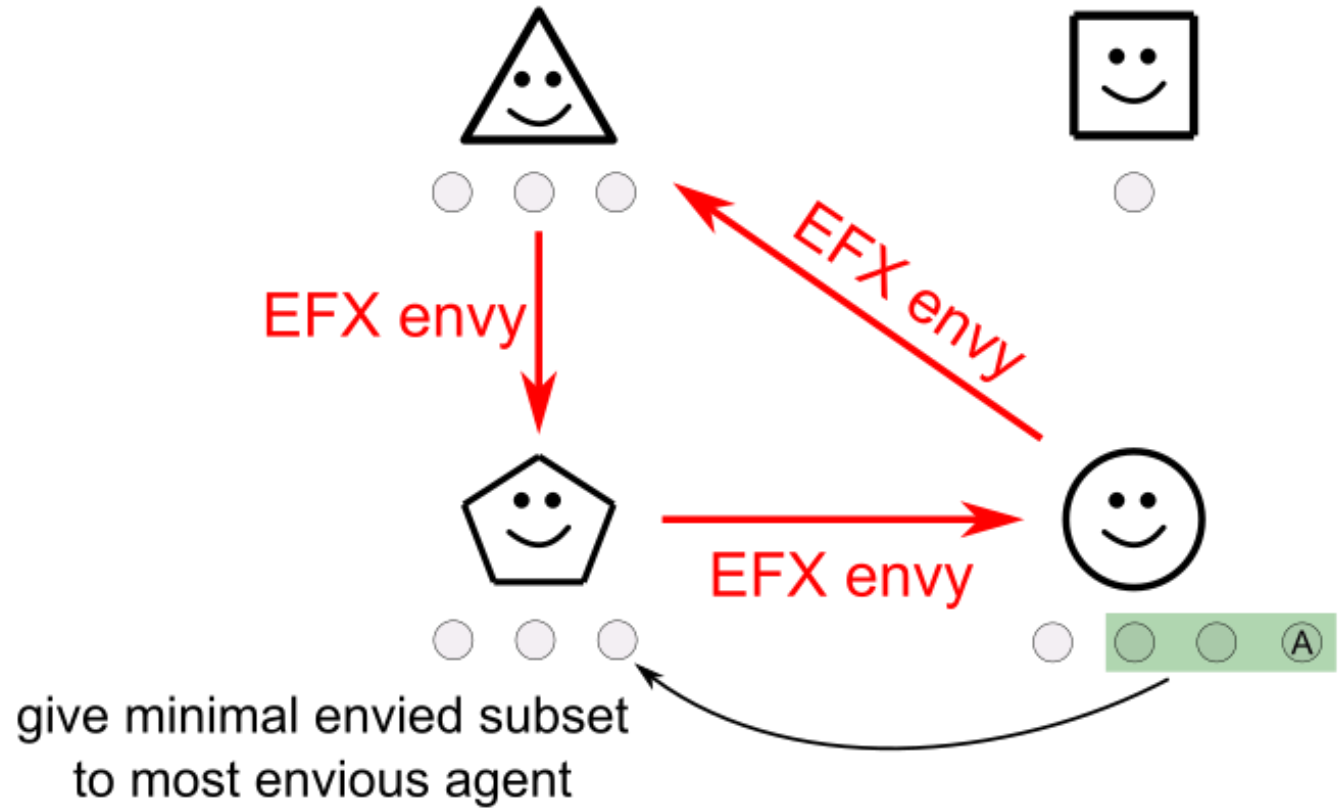

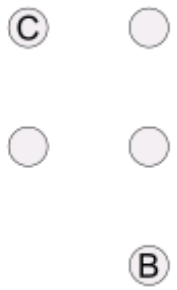

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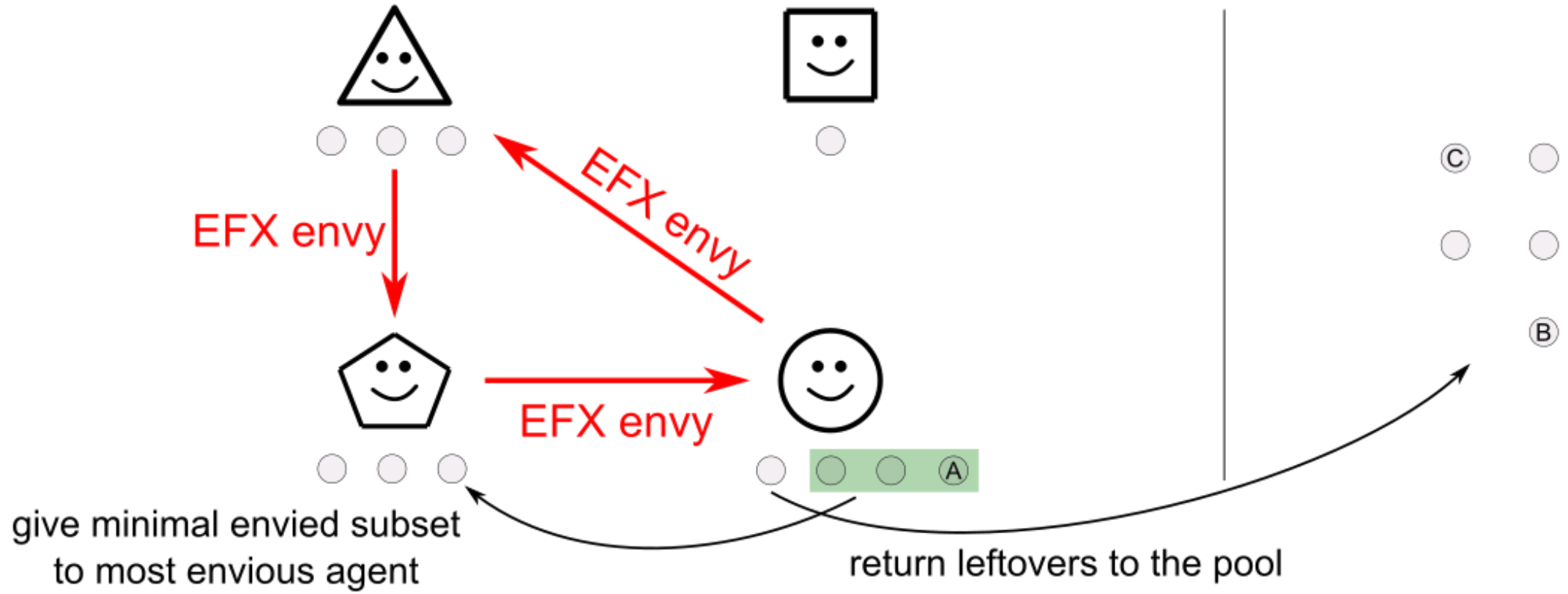

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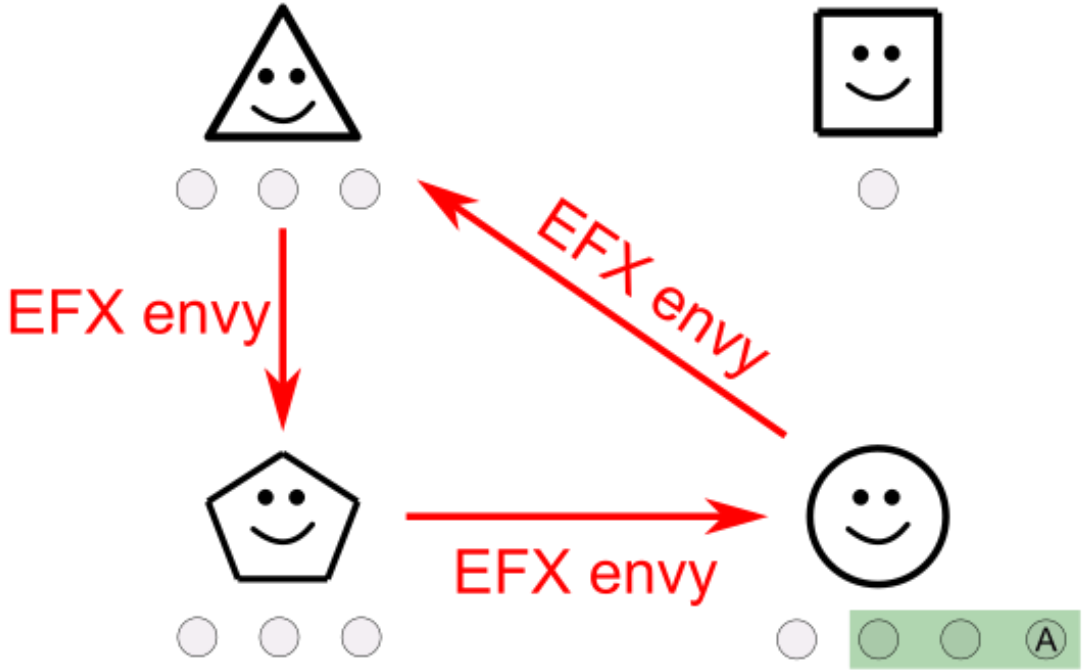

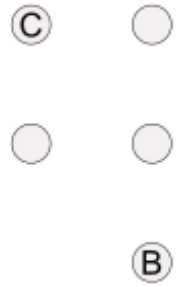

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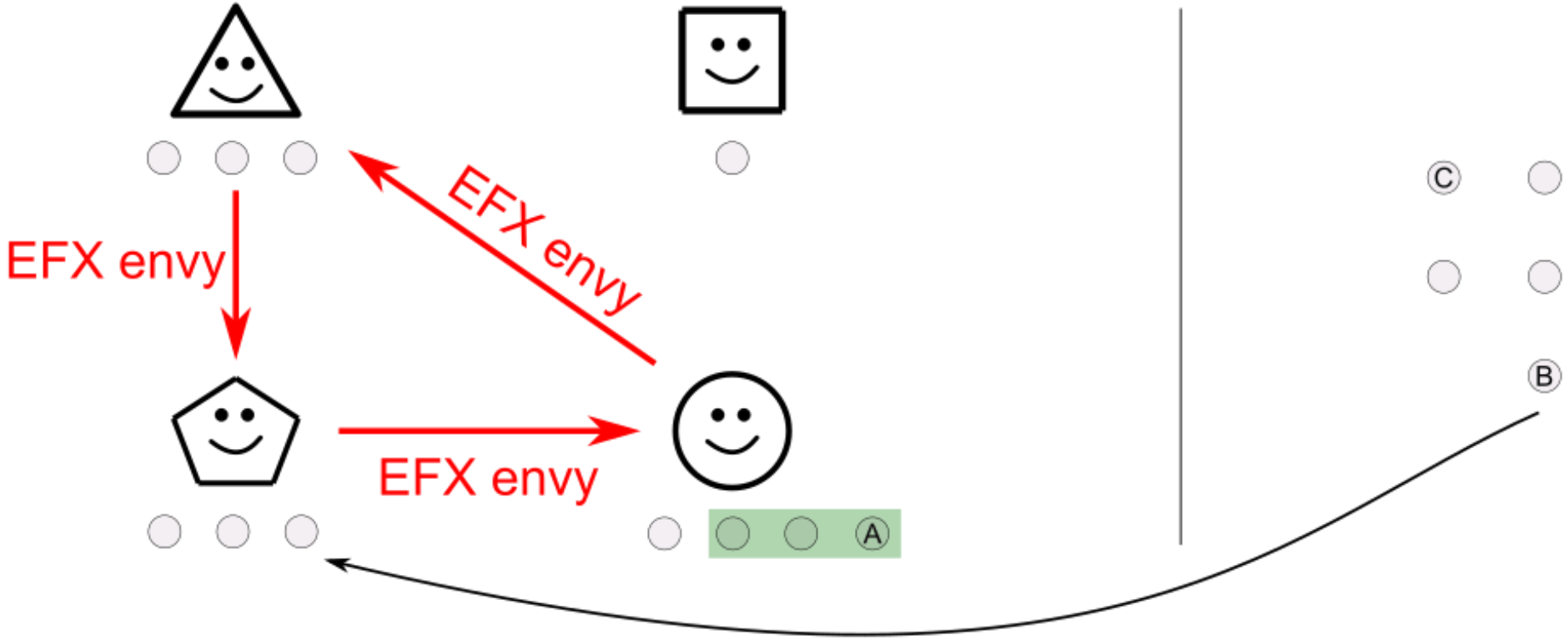

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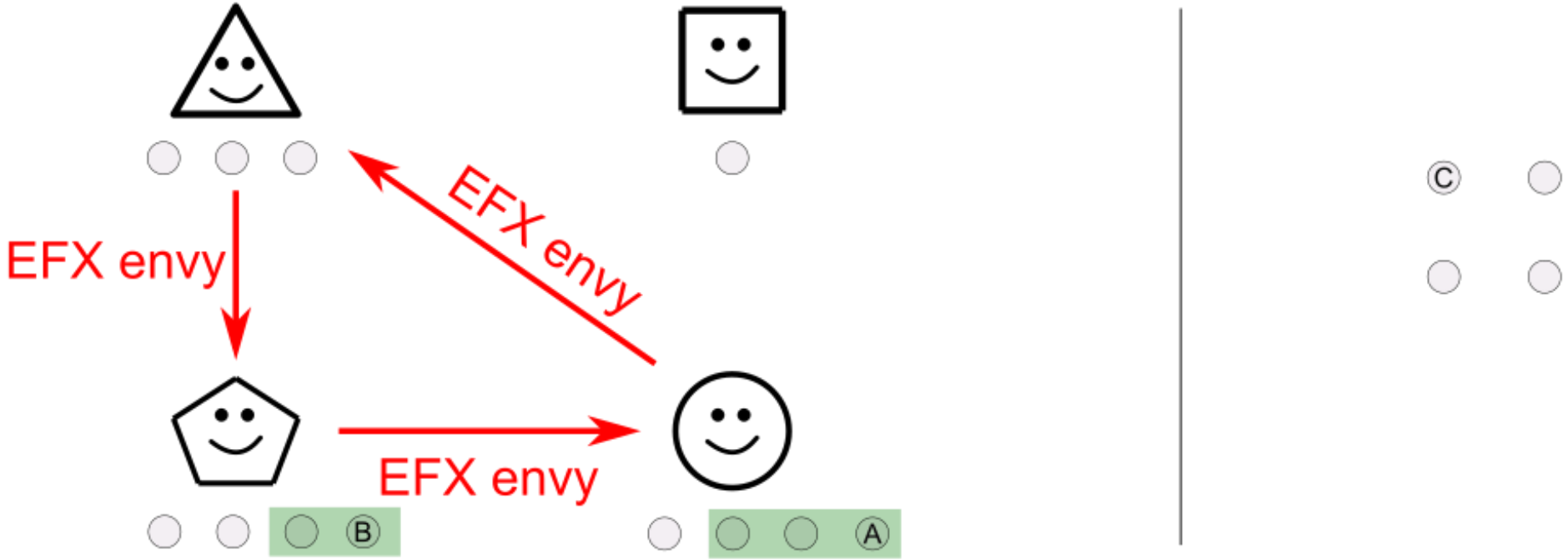

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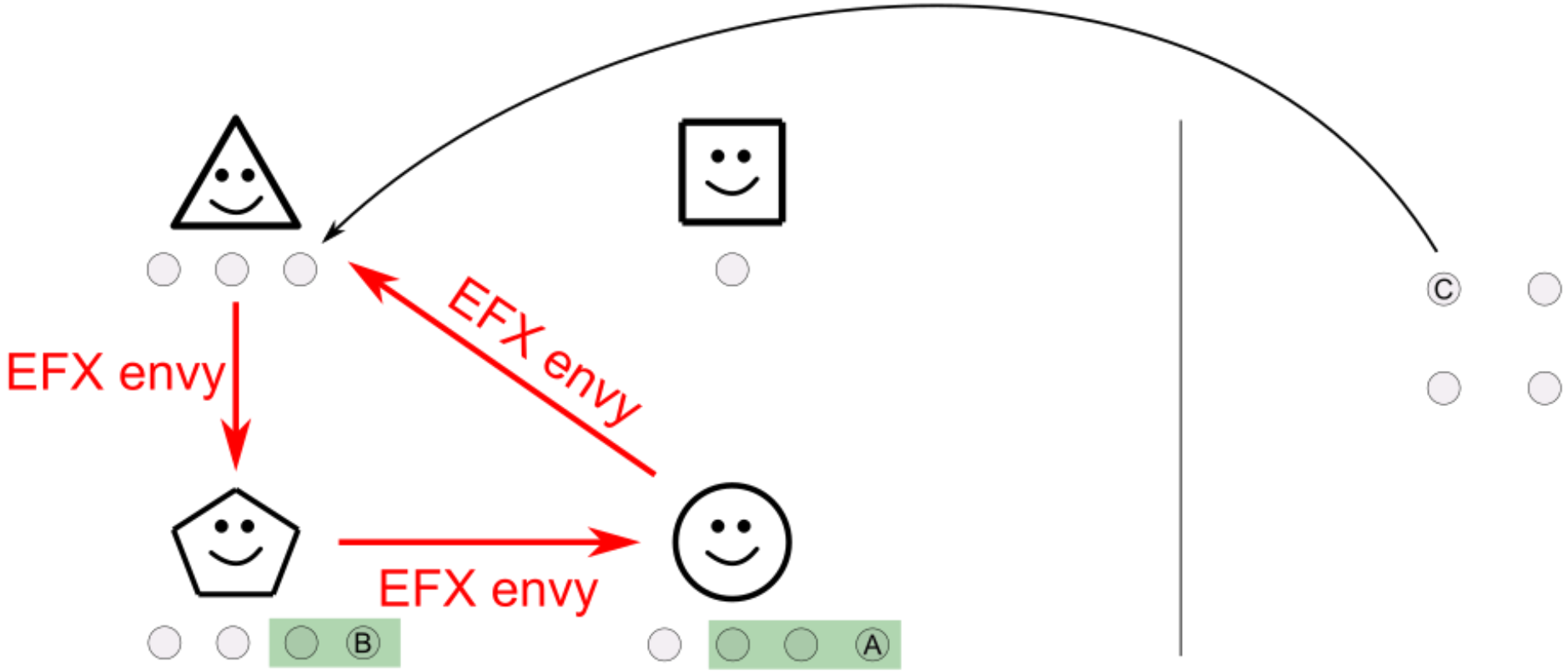

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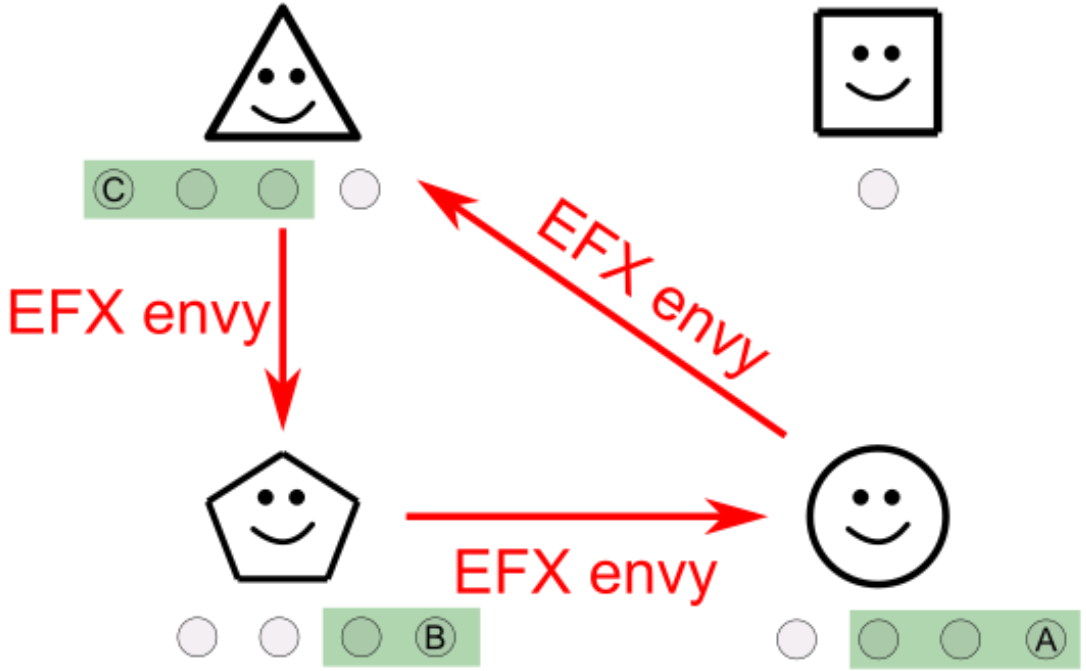

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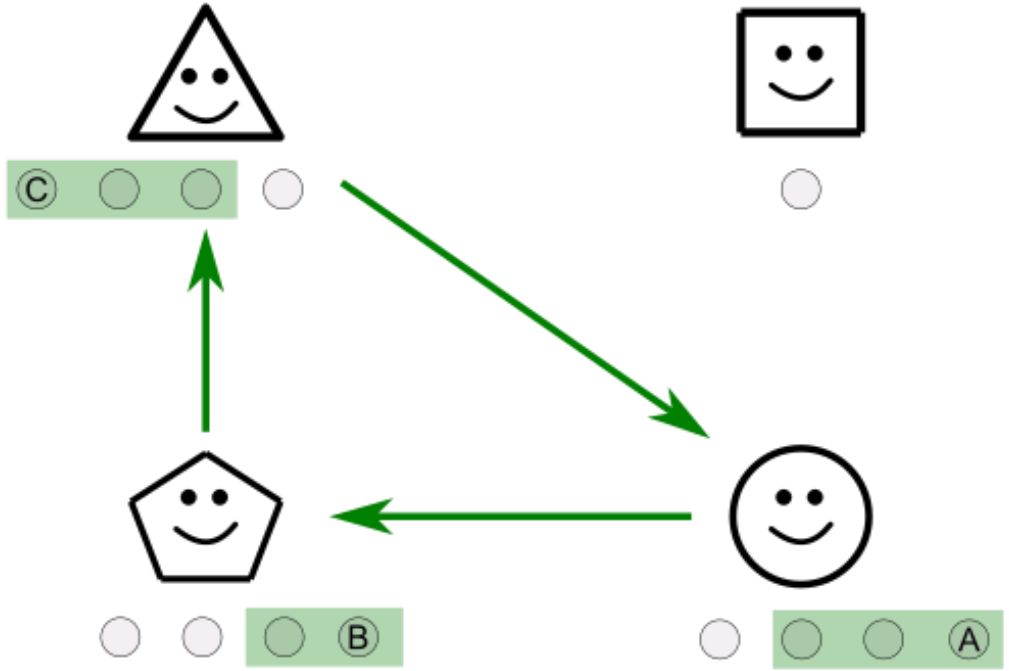

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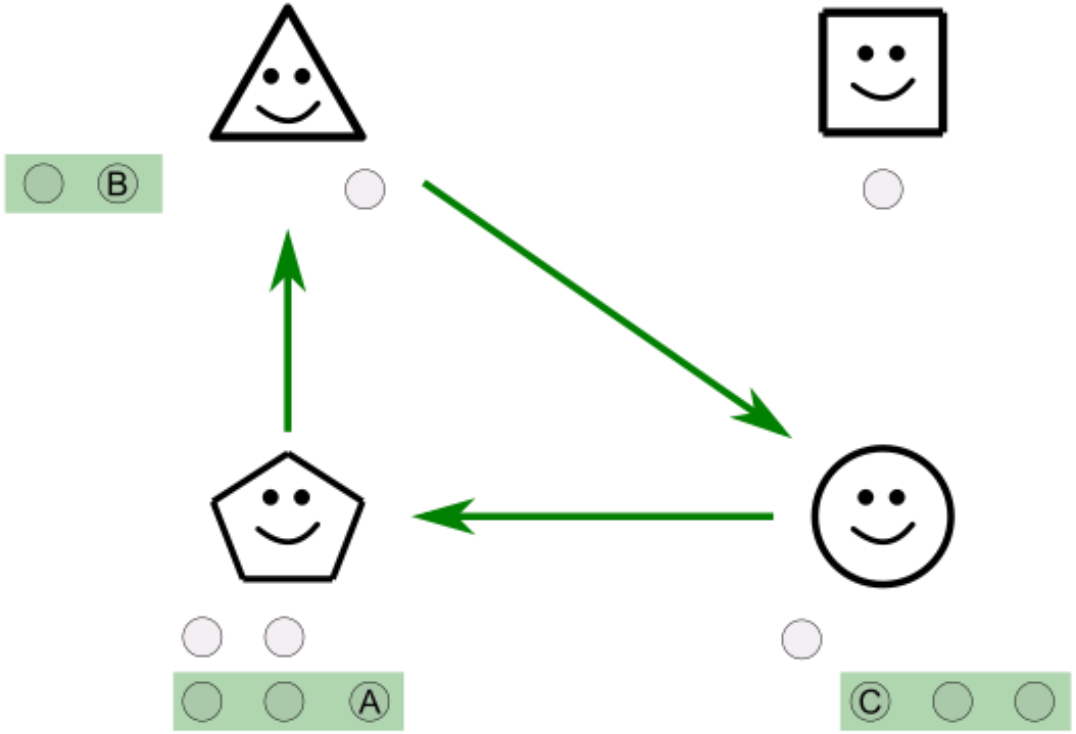

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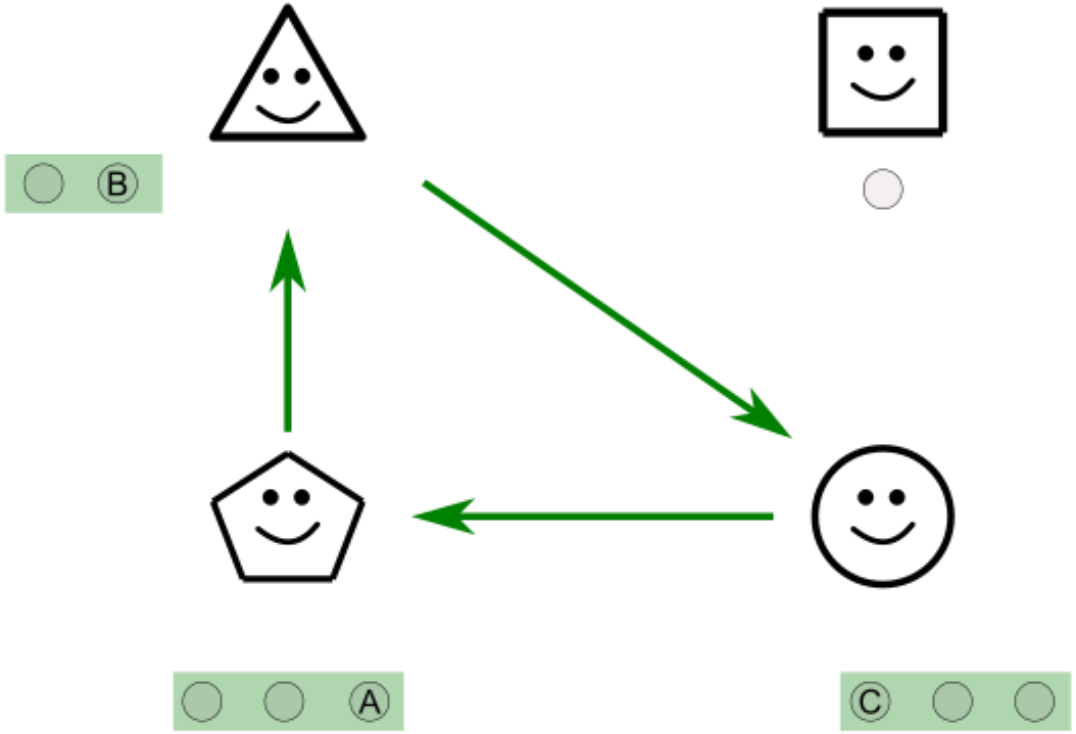

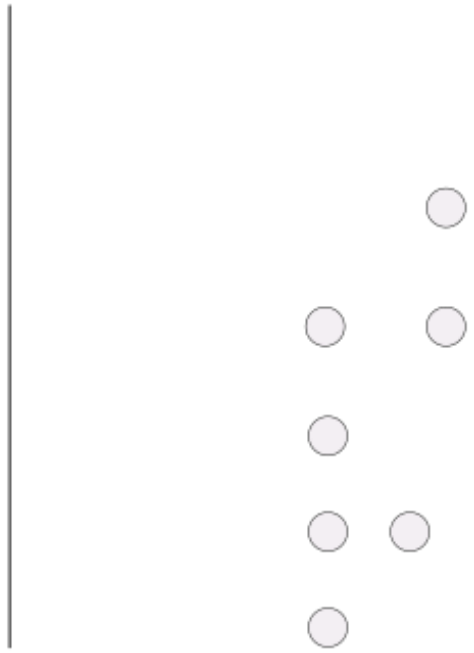

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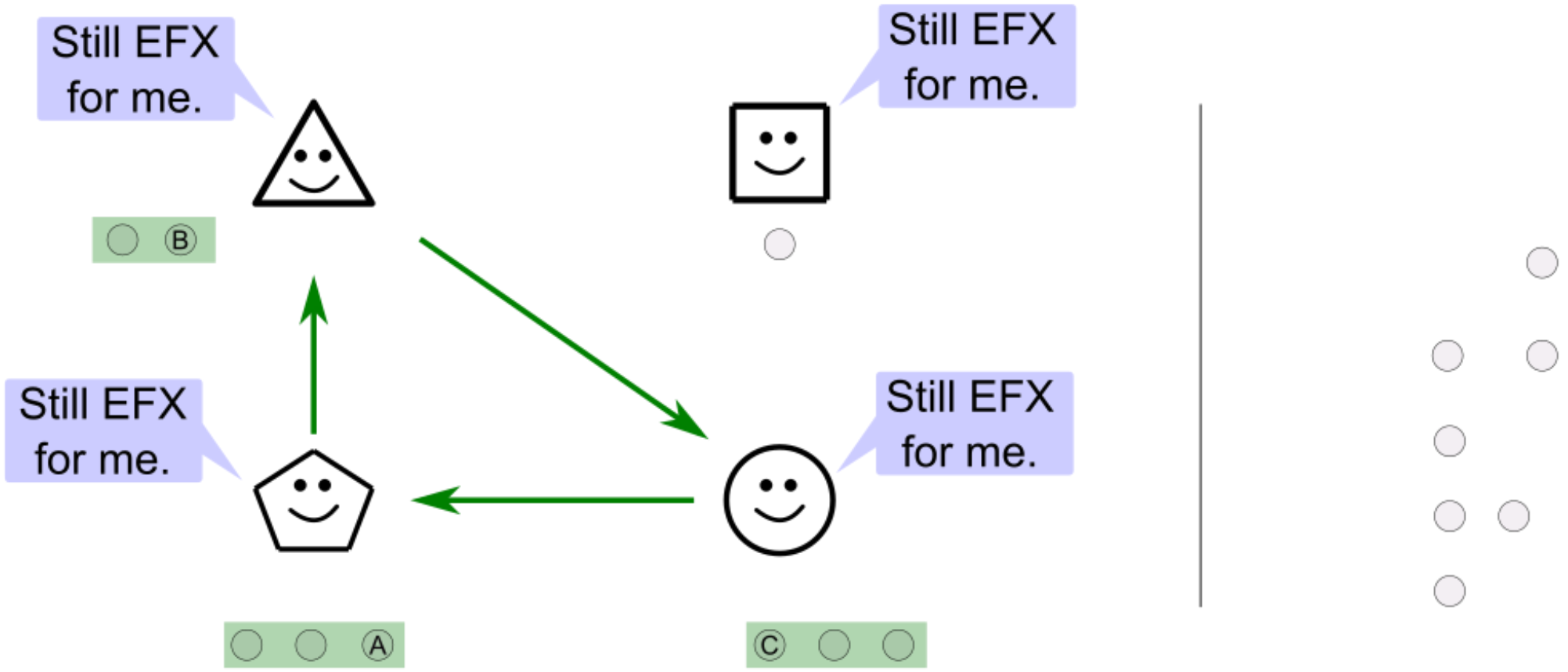

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If the algorithm terminates:

- EFX among main agents (invariant)
- No envy towards pool (Rule 2)
- $|P|<n$ (Rule 3)

$\Rightarrow$ EFX-with-charity

Achieving EFX-with-charity

[Chaudhury, Kavitha, Mehlhorn, and Sgouritsa, *SODA* 2020, *SICOMP* 2021]

Why does the algorithm terminate?

Achieving EFX-with-charity

[Chaudhury, Kavitha, Mehlhorn, and Sgouritsa, *SODA* 2020, *SICOMP* 2021]

Why does the algorithm terminate?

Rule 1: If an unallocated good can be given to an agent while maintaining EFX, then do so.

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Social welfare $\uparrow$, $|P| \updownarrow$

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Why does the algorithm terminate?

Potential function

$$\phi(A) = m \cdot \sum_i v_i(A_i) - |P|$$

$$m = \text{no. of goods}$$

EFX-with-charity in polynomial time

Next Time

Indivisible Chores

Quiz

Quiz

Construct an instance with two agents where no EFX allocation is Pareto optimal.